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PAPER

On the Integer Roots of Polynomial Equations

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Extended Abstract

Integer solutions of polynomial equations are very important in number theory. However, solutions of general polynomial equations of degree five or higher than five cannot be solved in radicals due to Abel-Ruffini theorem. Even in case of a quadratic equation, it is not easy at all to discard all integer solutions without knowing the coefficients of the independent variable. For example, the simple cubic $x^3 + 5x^2 + 12bx + q = 0$, where $(2, q) = 1$, has no integer roots. But, even in this case, it is not easy to justify this fact right away. The simple theorem is given in this paper which is capable of discarding integer roots of this equation at once. This theorem and the lemma [1] which was used in a previous paper are capable of discarding all integer roots of a special class of polynomial equations of any degree.

Solution of polynomial equations

First of all we consider simple statements with respect to the quadratic equation

$$ax^2 + bx + c = 0 \qquad (1)$$

The equation (1) has no integer roots if a, b, c all are odd or c is odd and a, b are of the same parity.

In general, the polynomial equation

$$a_0 x^n + a_1 x^{n-1} + ... + a_{n-1} x + a_n = 0,$$

has no integer roots if a_n is odd and $a_0, a_1, ..., a_{n-1}$ are even. Let us call this simple theorem, the theorem of odd parity. This follows at once since zero is even and integer roots of the equation are odd factor of a_n.

From (1) and the theorem of odd parity, we can deduce that $\Delta = b^2 - 4ac$ cannot be an integer when a, b, c all are odd.

Now consider the polynomial equation of the form

$$x^p + pbx - c^p = 0 \qquad (2)$$

It can be shown that this polynomial equation (2) has no integer roots for any odd prime p, where $(p, b) = (p, c) = 1$. By the theorem of odd parity, the equation has no integer roots when c and b are odd. If a is an integer root of the equation, then we must have

$$a^p - c^p + pba = 0$$

It is clear that $a^p - c^p \equiv 0(\mathrm{mod}\, p^2)$. This condition can't be satisfied since $(ba, p) = 1$ and hence the equation (2) cannot have integer roots.

The equation of the type

$$x^p + p^m bx - c^p - d^p p^{pm} = 0, \qquad (3)$$

where $m > 1$, $(cd, p) = 1$, has no integer roots.

If x is an integer root, $x^p - c^p \equiv 0(\mathrm{mod}\, p^m)$ and therefore $x - c \equiv 0(\mathrm{mod}\, p^{m-1})$ and we can write $x = c + p^{m-1} j$, where j is an integer co-prime to p.

Now, we can write

$$[(c + p^{m-1} j)^{p-1} + p^m b](c + p^{m-1} j) = c^p + d^p p^{pm}$$

and hence we must have

$$- p^{pm-p} j^p + d^p p^{pm} = 0$$

which implies p divides j. This is a contradiction and we deduce that the equation (3) has no integer roots.

References

[1] Piyadasa R.D.A., et.al. Simple Analytical Proofs of Three Fermat's Theorems, CMNSEM, Vol.2, No.3, (March 2011).