4.19 Analytical proof of Fermat's last theorem for n=4

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ABSTRACT

Fermat's last theorem for $n = 4$ is usually proved [1] using the famous mathematical tool of the method of infinite descent of Fermat. In this contribution, it will be shown that the parametric solution of the polynomial equation $d^4 = e^4 + g^4$, $(e, g) = 1$ can be obtained using a simple mathematical technique and thereby the proof of the theorem can be done, without depending on the sophisticated structure of primitive Pythagorean triples of Fermat[1] given by $x = 2lm, y = l^2 - m^2, z = l^2 + m^2$, where $l > m > 0$ and l, m are of opposite parity. The main objective of this contribution is to introduce a new simple mathematical technique which may be very useful in some other problems as well.

If the equation

$$d^4 = e^4 + g^4, \quad (e,g) = 1 \tag{1}$$

has a non-trivial integral solution for (x, y, z), then one of e, g is even and we can assume that d, e, g are positive. If g is even, $(d^2 - g^2)(d^2 + g^2) = e^4$ and terms in the brackets are co-prime and hence , one writes

$$d^2 + g^2 = x^4 \tag{2a}$$

$$d^2 - g^2 = y^4 \tag{2b}$$

From these two equations, we get

$$2d^2 = x^4 + y^4. \tag{2c}$$

Therefore $(x^2 - d)(x^2 + d) = (d - y^2)(d + y^2)$ and it is easy to deduce that terms in the brackets on the left-hand side or on the right-hand side of this equation may have only factor 2 in common since all numbers are odd and x, d, y are co-prime to one another. In the following, a new simple mathematical technique is used to obtain the parametric solution for x, y ,d, g from this single equation.

If $x^2 - d = d - y^2$, $2d = x^2 + y^2$ and therefore $4d^2 = x^4 + y^4 + 2x^2y^2$, which means $d^2 = x^2y^2$, and it leads to a contradiction since $(d, e) = 1$. Similarly we can easily show that $x^2 - d \neq d + y^2$. Now, let $(d - y^2) = \frac{a}{b}(x^2 - d)$,to obtain $x^2 - d = ba^{-1}(d - y^2)$, where $(a, b) = 1$ .Then $\frac{b}{a}(x^2 + d \ ) = (d + y^2)$, $x^2 + d = ab^{-1}(d + y^2)$ .Now, let us form the following two simultaneous equations,

$$x^2 - d = ba^{-1}(d - y^2) \tag{a}$$

$$x^2 + d = ab^{-1}(d + y^2) \tag{b}$$

to obtain,

$$2x^2ab = a^2(d + y^2) + b^2(d - y^2) \tag{3}$$

$$2abd = a^2(d+y^2) - b^2(d-y^2) = (a^2-b^2)d + (b^2+a^2)y^2 \qquad (4)$$

Since $(d,y)=1$, $b^2+a^2=dk$, where k has to be determined. Then , one easily obtains $2ab = a^2-b^2+y^2k$, $y^2 = \dfrac{2ab+b^2-a^2}{k}$, $d = \dfrac{a^2+b^2}{k}$.Now , from(3), it follows that

$$2abx^2 = (a^2+b^2)d + (a^2-b^2)y^2 = \frac{(a^2+b^2)^2 + (a^2-b^2)(2ab+b^2-a^2)}{k}$$

Hence, $x^2 = \dfrac{2ab+a^2-b^2}{k}$ and , from which it follows that

$$x^2y^2k^2 = 4a^2b^2 - (a^2-b^2)^2 \ , \quad (a^2-b^2)^2 + k^2x^2y^2 = 4a^2b^2 \qquad (5)$$

It is clear from (5) that a and b cannot be of opposite parity since then $k^2x^2y^2$ cannot be either odd or even. Hence a and b are both odd. and therefore $k^2=4$ or $4 \mid k^2$. Therefore $x^2y^2 = (4a^2b^2 - (a^2-b^2)^2)/4 = e^2, d = \dfrac{a^2+b^2}{2}$, $ab(a^2-b^2) = g^2$ as given below ,which is the parametric solution of the equation (1), where a,b are parameters.

Now, $x^2 - d = \dfrac{2ab+a^2-b^2-a^2-b^2}{k} = \dfrac{2a(a-b)}{k}$ and k is a factor of a^2+b^2 and if it is a factor of $a-b$, one deduces that k is 2 or a factor of a or b. Since $(a,b)=1$, we conclude that $k = 2$. Therefore $x^2y^2 = a^2b^2 - (a^2-b^2)^2/4$

Since $(x^2-y^2)(x^2+y^2) = x^4 - y^4 = 2g^2$, which follows from(2a),(2b), it is easy to deduce

$$ab(a^2-b^2) = g^2 \qquad (6)$$

Therefore $a,b,(a^2-b^2)$ should be perfect squares. Now, if $a=r^2, b=s^2$,then $r^4-s^4=t^2$ for some integers r,s,t. The famous and the only theorem that Fermat has proved is that there are no integers r,s,t satisfying $r^4-s^4=t^2$. Hence the Fermat's last theorem for $n=4$ can be deduced. It is quite interesting that applying the mathematical technique used in this contribution ,we have shown[2] very easily that the equation $r^4-s^4=t^2$ has no non- trivial integral solution for r,s,t ,and then the Fermat's last theorem for $n=4$ follows at once[1].

References

(1)Paulo Rebenboim, Fermat's Last theorem for amateurs, Springer , 1991

(2)W.M.J.L.P.Jayasighe, R.A.D.Piyadasa , to be published at the 9th Annual Research Symposium,University of Kelaniya,2008