An algorithmic method for the solution of the Lane Emden equation

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ABSTRACT

The Lane Emden Equation, which was first studied by the astro-physicist Emden is as follows

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + y^m = 0, \qquad m \in N$$

With boundary conditions, $y(0) = 1$, $y'(0) = 0$, the following solution satisfying the boundary conditions was obtained by J.R. Airey.

$$y = 1 - \frac{x^2}{3!} + m\frac{x^4}{5!} + (5m - 8m^2)\frac{x^6}{3.7!} + (70m - 183m^2 + 122m^3)\frac{x^8}{9.9!}$$
$$+ (3150m - 10805m^2 + 12642m^3 - 5032m^4)\frac{x^{10}}{45.11!} + ...$$

We extend this solution up to the term in x^{12}.

Leibniz's formula was used to obtain the following recurrence relation (RR)

$$y_{n+2} = -\frac{n+1}{n+3}(y^m)_n$$

Here $y_n = \frac{d^n y}{dx^n}(0)$ and $(y^m)_n = \frac{d^n(y^m)}{dx^n}(0)$

Now a Taylor's expansion about x=0, will produce the solution. Computational difficulties arise in evaluating $(y^m)_n$. We have discovered the following algorithm, which will make the computation less tedious.

$$(y'')_n = \sum {}^nC_r \frac{n!}{S_1!S_2!\ldots\ldots\ldots S_r!} N(S_1,S_2,\ldots\ldots\ldots S_r)y_{s_1}y_{s_2}\ldots\ldots\ldots y_{s_r}$$

Where $N(S_1,S_2\ldots\ldots\ldots S_r)$ is the number of possible arrangements of $S_1,S_2\ldots\ldots\ldots S_r$ such that $S_1 + S_2 +\ldots\ldots\ldots +S_r = n$, $S_r \geq S_{r-1}\ldots\ldots\ldots \geq S_1$. The summation is over at all the distinct partitions of n.

As the equation is unchanged when -x is substituted for x the expansion contains only even powers of x and the derivatives of y of odd order evaluated at x=0 are all zero. From this it follows that only even partitions of even integers have to be considered.

For the prescribed boundary conditions, it can be shown by other methods that the solutions, given by the following particular m are

$$m=0: \quad y=1-\frac{x^2}{6}; \quad m=1: \ y=\frac{\sin x}{x}; \quad m=5: y=(1+\frac{x^2}{3})^{-\frac{1}{2}}$$

These are very useful, since they can be used to verify the expansion obtained by our method. The Taylor expansion of the solution is given by putting n=0 in the recurrence relation (RR),

$$y_2 = -\frac{1}{3}$$

n=2 gives, $y_4 = -\frac{3}{5}(y'')_2$

Using the algorithm (even partitions of 2 are (2))

$$(y'')_2 = {}^mC_2 \frac{2!}{2!} N(2)y_2$$

$$= my_2$$

$$= -\frac{m}{3}$$

Hence, using the RR, $\quad y_4 = -\frac{3}{5}(-\frac{m}{3}) = \frac{m}{5}$

n=4 gives, $\quad y_6 = -\frac{5}{7}(y''')_4$

Using the algorithm (even partitions of 4 are (2,2), (4))

$$(y''')_4 = {}^mC_2 \frac{4!}{2!2!} N(2,2) y_2 y_2 + {}^mC_1 \frac{4!}{4!} N(4) y_4,$$

$$= \frac{m(m-1)}{2!} \frac{4!}{2!2!} (-\frac{1}{3})(-\frac{1}{3}) + m \frac{4!}{4!} \frac{m}{5}; \qquad N(2,2) = 1, \quad N(4) = 1,$$

$$= \frac{m}{15}(8m - 5)$$

From the RR for n=4 we have,

$$y_6 = -\frac{5}{7}(y''')_4$$

$$= -\frac{5}{7}\frac{m}{15}(8m - 5)$$

$$= \frac{1}{21}(5m - 8m^2)$$

Using the algorithm (even partitions of 6 are (2,2,2), (2,4), (6))

$$(y''')_6 = {}^mC_3 \frac{6!}{2!2!} N(2,2,2) y_2 y_2 y_2 + {}^mC_2 \frac{6!}{2!4!} N(2,4) y_2 y_4 + {}^mC_1 \frac{6!}{6!} N(6) y_6,$$

$$N(2,2,2) = 1, \qquad N(2,4) = 2!, \qquad N(6) = 1$$

From the RR for n=6, $y_8 = -\frac{7}{5}(y''')_6$

It is clear that this procedure can be followed to evaluate any y_{2n}. A computer program and a package like the Mathematica can be used effectively to obtain a Taylor expansion up to any power of x, with the use of RR and the algorithm.