New Theorem on Primitive Pythagorean Triples

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As a result of our survey on primitive Pythagorean triples, we were able to prove the following theorem:
All primitive Pythagorean triples can be generated by almost one parameter α, satisfying $\alpha > \sqrt{2} + 1$. Furthermore, α is either an integer or of the form $\alpha = \frac{\gamma}{\eta}$ where γ and η (> 1) are relatively prime numbers.
The proof of the theorem can be briefly outlined as follows:
Taking $z = y + p$ for some $p \geq 1$, $z^2 = y^2 + x^2$ can be put into the form

$$\left(1 + \frac{p}{y}\right)^2 = 1 + \left(\frac{x}{y}\right)^2$$

If $\alpha = \frac{x}{p}$, then the above equation can be put into the form

$$(1+\beta)^2 = 1 + \alpha^2 \beta^2 \quad \text{...} \quad (1),$$

where $\frac{1}{\beta} = \frac{\alpha^2 - 1}{2}$. Then the above equation can be reduced into

$$\left(1 + \frac{\alpha^2 - 1}{2}\right)^2 = \left(\frac{\alpha^2 - 1}{2}\right)^2 + \alpha^2 .$$

In order to generate primitive triples, the above equation has to be multiplied by 4 if α is even and η^4 if $\alpha = \frac{\gamma}{\eta}$. Now we are able to generate all the primitive Pythagorean triples if α satisfies the conditions of our theorem and $\frac{\alpha^2 - 1}{2}$ is reduced to cancel 2 in the denominator whenever necessary. The condition $\alpha > \sqrt{2} + 1$ and α is either integer or of the form $\alpha = \frac{\gamma}{\eta}$ $(\eta > 1)$ with γ and η are relatively prime odd be imposed after a careful study of the equation . In conclusion, an algorithm can be developed to determine p and y so that $((y+p), y, x)$ is a primitive Pythagorean triple in the order $x < y < y + p$ for given x. A new theorem on primitive Pythagorean triples is found and it may be useful in understanding the Fermat's Last Theorem.

Key Words: Primitive pythagorean triples, Fermat's Last Theorem.

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