Null Geodesics in de-Sitter Universe

J.A.N.K. Jayakody[1], L.N.K. de Silva[2]

Department of Physics[1], University of Kelaniya

Department of Mathematics[2] University of Kelaniya

ABSTRACT

Consider the de-Sitter universe given by

$$ds^2 = c^2dt^2 - e^{2Ht}\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]^{[1]}, \quad (1)$$

which involves the cosmological constant.

Here H is the Hubble constant and $H = \sqrt{\frac{\Lambda}{3}}$; Λ is the cosmological constant.

For null-geodesics $ds^2 = 0$.

$$\text{Let} \quad d\mu^2 = c^2dt^2 - e^{2Ht}\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \quad (2)$$

Dividing both sides of (2) by the element $d\mu^2$,

$$L = c^2\left(\frac{dt}{d\mu}\right)^2 - e^{2Ht}\left(\frac{dr}{d\mu}\right)^2 - e^{2Ht}r^2\left[\left(\frac{d\theta}{d\mu}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\mu}\right)^2\right] = 0. \quad (3)$$

The equations of the null-geodesics can be written by using the Euler-Lagrange's equations

$$\frac{d}{d\mu}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0, \quad (4)$$

where $x = (t, r, \theta, \phi)$ and dot denotes differentiation with respect to μ.

Considering the t coordinate,

$$\frac{d}{d\mu}\left[2c^2\left(\frac{dt}{d\mu}\right)\right] + 2He^{2Ht}\left(\frac{dr}{d\mu}\right)^2 + 2He^{2Ht}r^2\left[\left(\frac{d\theta}{d\mu}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\mu}\right)^2\right] = 0. \quad (5)$$

Considering the θ coordinate,

$$\frac{d}{d\mu}\left[-2e^{2Ht}r^2\left(\frac{d\theta}{d\mu}\right)\right] + 2e^{2Ht}r^2\sin\theta\cos\theta\left(\frac{d\phi}{d\mu}\right)^2 = 0. \quad (6)$$

Considering the ϕ coordinate,

$$\frac{d}{d\mu}\left[-2e^{2Ht}r^2\sin^2\theta\left(\frac{d\phi}{d\mu}\right)\right]=0\,. \qquad (7)$$

Since $ds=0$, for null-geodesics we also have

$$c^2\left(\frac{dt}{d\mu}\right)^2-e^{2Ht}\left(\frac{dr}{d\mu}\right)^2-e^{2Ht}r^2\left[\left(\frac{d\theta}{d\mu}\right)^2+\sin^2\theta\left(\frac{d\phi}{d\mu}\right)^2\right]=0 \qquad (8)$$

These are the null-geodesics equations of the de-Sitter space-time.

Equation (6) is satisfied by $\theta=\frac{\pi}{2}$.

From (5),

$$\frac{d}{d\mu}\left[2c^2\left(\frac{dt}{d\mu}\right)\right]+2He^{2Ht}\left(\frac{dr}{d\mu}\right)^2+2He^{2Ht}r^2\left(\frac{d\phi}{d\mu}\right)^2=0 \qquad (9)$$

From (7),

$$\frac{d}{d\mu}\left[-2e^{2Ht}r^2\left(\frac{d\phi}{d\mu}\right)\right]=0$$

$$-2e^{2Ht}r^2\left(\frac{d\phi}{d\mu}\right)=c_1 \qquad ; \quad \text{where } c_1 \text{ is a constant}$$

$$d\mu=\frac{-2e^{2Ht}r^2}{c_1}d\phi \qquad (10)$$

From (8),

$$c^2\left(\frac{dt}{d\mu}\right)^2-e^{2Ht}\left(\frac{dr}{d\mu}\right)^2-e^{2Ht}r^2\left(\frac{d\phi}{d\mu}\right)^2=0 \qquad (11)$$

From (9) and (10), we obtain,

$$\left(\frac{dr}{d\phi}\right)^2=-\left(\frac{d^2t}{d\mu^2}+\frac{Hc_1^{\,2}}{4c^2e^{2Ht}r^2}\right)\frac{4c^2e^{2Ht}r^4}{Hc_1^{\,2}} \qquad (12)$$

From (10) and (11), we have,

$$\left(\frac{dr}{d\phi}\right)^2 = \left[\left(\frac{dt}{d\mu}\right)^2 - \frac{c_1^2}{4c^2e^{2Ht}r^2}\right]\frac{4c^2e^{2Ht}r^4}{c_1^2} \qquad (13)$$

(12) and (13) give,

$$\frac{1}{H}\left(\frac{d^2t}{d\mu^2}\right)+\left(\frac{dt}{d\mu}\right)^2 = 0$$

Let $\frac{dt}{d\mu} = p$. Then $\frac{d^2t}{d\mu^2} = p\frac{dp}{dt}$

Let $p^2 = v$. Then $2p\frac{dp}{dt} = \frac{dv}{dt}$ and finally we have,

$$\frac{dt}{d\mu} = \sqrt{e^{-(2Ht+c_2)}} \quad ; \quad \text{where } c_2 \text{ is a constant} \qquad (14)$$

From (13) and (14), substituting $r = \frac{1}{u}$ we obtain

$$\frac{du}{d\phi} = \sqrt{\frac{4c^2e^{-c_2}}{c_1^2} - u^2}$$

$$\sin^{-1}\frac{u}{\sqrt{\frac{4c^2e^{-c_2}}{c_1^2}}} = \phi + c_3 \quad ; \quad \text{where } c_3 \text{ is a constant.}$$

$$u = \frac{2c}{c_1}\sqrt{e^{-c_2}}\sin(\phi + c_3)$$

Thus the equation of a null geodesic in de Sitter universe is given by the above expression which agrees with the equation $u = \left(\frac{E^2}{h^2} + \frac{\Lambda}{3}\right)^{\frac{1}{2}}\sin(\phi + A)$ [2] we had in the case of Schwarzschild - de Sitter metric with $m = 0$, after identifying the corresponding constants.

References

[1] Narlikar, J.V., (1993) *'Introduction to Cosmology'*, Cambridge University Press.

[2] Jayakody, J.A.N.K., de Silva, L.N.K. *'Path of a light ray near a body with cosmological constant'*, 10th Annual Research Symposium 2009, University of Kelaniya (2009).