mathematics

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Solutions of Sturm-Liouville Problems

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Received: 27 October 2020; Accepted: 18 November 2020; Published: 20 November 2020

Abstract: This paper further improves the Lie group method with Magnus expansion proposed in a previous paper by the authors, to solve some types of direct singular Sturm–Liouville problems. Next, a concrete implementation to the inverse Sturm–Liouville problem algorithm proposed by Barcilon (1974) is provided. Furthermore, computational feasibility and applicability of this algorithm to solve inverse Sturm–Liouville problems of higher order (for $n = 2, 4$) are verified successfully. It is observed that the method is successful even in the presence of significant noise, provided that the assumptions of the algorithm are satisfied. In conclusion, this work provides a method that can be adapted successfully for solving a direct (regular/singular) or inverse Sturm–Liouville problem (SLP) of an arbitrary order with arbitrary boundary conditions.

Keywords: Sturm–Liouville problems of higher order; singular Sturm–Liouville problems; inverse Sturm–Liouville problems

MSC: 34L16; 65L09; 34B24

1. Introduction

The inverse Sturm–Liouville theory was originated in 1929 by Ambarzumian [1] and further developed in [2–6].

The $2m$th order, nonsingular, self-adjoint eigenvalue problem (EVP) or Sturm–Liouville problem (SLP) is given by:

$$\begin{aligned}&(-1)^m(p_0(x)y^{(m)})^{(m)} + (-1)^{m-1}(p_1(x)y^{(m-1)})^{(m-1)}\\&+\ldots+(p_{m-2}(x)y'')'' - (p_{m-1}(x)y')' + p_m(x)y = \lambda w(x)y, \qquad a < x < b\end{aligned} \tag{1}$$

together with some boundary conditions at a and b, the functions p_k, $(0 \le k \le m)$, and $w(x)$ being continuous on the finite closed interval $[a, b]$, and p_0 having a continuous derivative. In the inverse SLP, the coefficient functions p_k, $(0 \le k \le m)$ need to be reconstructed, given suitable valid spectral data.

For a discussion of analytical methods and numerical methods for inverse SLP, see [7,8], respectively. Iterative methods [9,10], Rayleigh–Ritz method [11], finite difference approximation [12], Quasi-Newton method [13], shooting method [14], interval Newton's method [15], finite-difference method [16], boundary value methods [17–20], Numerov's method [21–23], least-squares functional [24], generalized Rundell–Sacks algorithm [25,26], spectral mappings [27], Lie-group estimation method [28], Broyden method [29,30], decent flow methods [31], modified Numerov's method [32], Newton-type method [33], Fourier–Legendre series [34], and Chebyshev polynomials [35] are of particular importance among the existing methods to solve inverse SLP.

Numerical algorithms to solve the inverse fourth-order Sturm–Liouville problem (FSLP) are proposed in [36–38].