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Threshold Net Profit Condition in Predicting the Insurer's Probability of Ruin

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ABSTRACT

Purpose: An insurer is technically ruined when its surplus falls below a specified level that is less than a defined benchmark. In ruin analysis, the classical Lundberg's model describes the claim's process of an insurer where the claim size and the inter-arrival times are independent of each other, but this may not be reasonable enough since it is complex to express the adjustment coefficient in terms of the distributions of claim sizes and inter-arrival times. It is, therefore, reasonable to compare the ruin probability of insurer's portfolio under Tijim's approximation in order to determine the level at which the insurer could survive. The objectives of this study are to estimate the adjustment coefficient using moment generating function, confirm when the net profit condition is violated and construct model for the exponential parameter of the Tijim's model.

Design/Methodology/Approach: We compare ruin probabilities Lundberg's and Tijim's frameworks under Gamma claims.

Findings: Computational evidence from the results reveals that Tijim's approximation is comparatively lower than Lundberg's upper bound and, therefore, represents an improvement. The empirical analysis suggests that the insurer should avoid initial reserve below 1,800,000.00. From tables 2 and 3, at any level of the initial capital u both models seem not converging to zero very fast. Within the interval $1000000 \leq u \leq 1800000$, the ruin probability is trivially confirming that the net profit condition $E(X) - \alpha < 0$ is violated.

Originality: This paper has improved the Tijim's estimation analytically as demonstrated in our empirical analysis.

KEYWORDS

Tijim's ruin, Gamma distribution, initial capital, survival probability, safety loading, ruin probability

JEL CLASSIFICATION

C13; C15; G52

I. Introduction

Ruin theory reflects the probability that an insurer assumes at a specified time horizon with initial capital and premiums received such that the major tool governing the theory is built on the surplus process which defines a stream of cash flows of the insurer. The initial capital is to buffer against insurance risk where premium collected may not be adequate to indemnify future losses.

The regulation of insurance sector in form of risk management strategy is anchored on the solvency guidelines enshrined in the Basel Accord for banking framework. This regulation is designed for the protection of the insured schemes by enforcing adequate capitalization for the insurers to create an enabling financially healthy environment. The solvency framework is built on three pillars: Pillar I is built on the required capitalization in order to protect an insurer from the likelihood of insolvency and to sustain the appropriate technical provisions. Pillar II is built on expert's opinions and independent judgment on the degree of risk assessment for solvency determination purposes to improve insurers' risk management techniques. Pillar III ensures the disclosure of information to the public to enhance market discipline. Therefore, ruin theory defines the amount of the required start-up capital for a specified probability level of insolvency.

It is of utmost importance that insurers predict their cash requirements precisely. However, finding an optimal balance

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between the capital requirements and ruin probability in a portfolio of insurance is very challenging and the inclusion of complex insurance products has not made it easier. Consequently, this paper is constructed in the light of ruin probability where the modelling of surplus process plays fundamental role in modelling ruin.

The goal of this study is to investigate the condition which violates the net premium condition in predicting the insurer's probability of ruin under the frameworks of two classical Lundberg and Tijim's ruin models. The general non-ruin integral equation based on monotone convergence theorem is first established. To achieve our investigation and as part of our contributions, the adjustment coefficient is numerically estimated using the moment generating function, and the exponential parameter of the Tijim's approximation is constructed as an extension. This accounts for the reason why the numerical technique is adopted in modelling underwriting loss through random variable. To enable us compare these ruin models, it is necessary to use the appropriate loss distribution. Since we cannot compute the exact ruin based on our data, we assume that the true distribution of the claim size follows gamma distribution.

In the non-life insurance underwriting business, a core issue is to address the risk evolving in the claim arrival and the severity of claim trend through a defined random loss function. This is a technical actuarial problem within the underwriting domain associated with the determination of the requisite capital to avoid insolvencies. Inadvertently, the risk in the claim arrival is addressed using a counting process while a continuous density function is adopted for modelling the severity of the claim. This is achieved with the goal of assessing underwriting uncertainties meant to determine the basis over which the adequacy of actuarial functions connected with pricing of insurance products, safety loading, ruin probability and reserves depend.

The insurance industry is a useful integral part of the Nigerian economy since it provides a means of spreading financial losses evolving out of uncertainties across a large number of insured bases. Probability of ultimate ruin is a useful device to measure financial risk of an insurer's portfolio eliciting powerful information on the probability of insurer's insolvencies and serving as an initial signal to the insurers. The time to ruin, the deficit at the time of ruin and the surplus prior to ruin are core indicators to address issues connected with insolvencies of the insurer. Consequently, the rationale behind the identified indicators is to advise the insurer to take precautions in terms of the reserve base required to avoid ruinous conditions. Specifically, the underlying goal of the deficit at the time of ruin is to advise on the mode of recovery where the insurer inadvertently falls into ruinous conditions.

II. Literature Review and Hypotheses

Avram, Biard, Dutang, Loisel and Rabehasaina (2014) addressed the asymptotic behavior of ruin probability based on mixing properties of claim distribution and adopted the matrix exponential estimation technique to compute ruin probabilities.

Furthermore, Koucha (2016) carried out estimations of ruin probabilities in infinite time under lévy process. The Cramér-Lundberg model of the collective risk process was adopted for the perturbed model by embedding a Lévy (α-stabled) process to the compound Poisson process. The study presented alternative estimation methodologies employed for the perturbed model in continuous time.

Zhou Sakhanenko and Guo (2020) investigated the ruin probabilities of non-homogeneous risk models. The author applied the martingale method to obtain Lundberg's inequalities under weak assumptions.

Loke and Thomann (2018) employed numerical method to investigate ruin probability in the dual risk model with risk-free investments. The author used the collocation method to estimate the ruin probability for any gain distribution by solving the underlying integral equation. Euphasio, Carvalho (2022) simulated Lundberg's process using the Monte Carlo technique by modifying some probability distributions to the severity of the compound Poisson process.

Leung (2021) attempted the solution of the equation of ruin probabilities in the infinite continuous time using the Fuorier transform and complex functions. The author obtained solutions as complex integrals in a form that could be numerically solved through the inverse Fourier transform.

Employing the theory of recurrence sequence, Santana & Luis (2023) obtained a new expression for the ultimate ruin probability in the classical Cramer-Lundberg's process where claims are assumed to follow a finite mixture of nErlang distribution. The author's technique reduces the issue of computing the ruin probabilities to the formulation of a characteristic polynomial.

Dickson and Willmot (2005) examined the Lundberg's fundamental equation

$$\lambda + \beta - Cs = \lambda \int_0^{\infty} e^{-sz} f(z)\, dz \qquad (1)$$

where β is a non-negative parameter. In their solution, the authors considered the equation

$$\varphi_\eta(u) = \boldsymbol{E}\left(e^{-\beta T_u} I(T_u < \infty)\right) \qquad (1a)$$

where

$$I(T_u < \infty) = \begin{cases} 1 & if \quad {}_u < \infty \\ 0 & if \quad otherwise \end{cases} \qquad (1b)$$

However, Gerber and Shiu (1998) provided a unique solution $\eta(\beta) = \beta$ to the equation.

Gerber and Dufresne (1991) carried out extensive work on the extension of the classical surplus process by including a diffusion term in an attempt to consider an extra risk of aggregate claims and premium income. The authors established the defective renewal equation as an extended renewal process, and consequently, this has become the benchmark of renewal theory applicable in obtaining ruin probability, particularly in mixtures of exponential distributions.

Gaps of the Literature

It has been observed that extant studies which incorporate the asymptotic properties of both Lundberg's upper bound and Tijims approximations are relatively scarce. Specifically, these extant literature hardly investigate the behavior of ruin under Tijim's framework. A common limitation of all these works is that none of these studies has justified that Tijim's approximation is an improvement over Lunberg's ruin upper bound. Moreover, it is further observed that it is not within the scope of these works to construct model for the exponential parameter of the Tijim's model. Consequently, this study investigates the improvement of Tijim's approximation over Lundberg's model under the assumption of Gamma claims distribution.

Cramer's ruin model is examined in its raw form with an emphasis on the asymptotics of ruin probabilities in the regime that the insurer's initial capital progressively becomes very large. Consider the surplus process.

$$U(\xi) = u + \boldsymbol{C}\xi - S(\xi); \quad u = U(0) \geq 0 \qquad (1c)$$

where $U(\xi)$ is the surplus at time $\xi > 0$ with an initial capital $U(0)$

$$ruin = \boldsymbol{P}\big(u + \boldsymbol{C}\xi < S(\xi)\big); \quad \xi > 0 \qquad (1d)$$

$\boldsymbol{C}$ is the gross risk premium rate received per unit time

$$\{S(\xi)\}_{\xi>0} = \begin{cases} \sum_{j=1}^{M(\xi)} z_j & M(\xi) > 0 \\ 0 & M(\xi) = 0 \end{cases} \qquad (1e)$$

is the aggregate claim paid during the interval $]0, \xi]$ and defining the cumulative amounts of claims up to time ξ. It clearly defines a convolution between severity and frequency of losses. The risk process input variables specified in the surplus process is to conduct the dynamics of an insurer's essential operations. Consequently, in view of the premium received and the claims paid, Lundberg's model is adequate to assess the reserve base of underwriting risk.

$M(\xi)$ is a set of integers counting the number of claim paid till time ξ and z_j represents the jth claim where

$$E(Z^j) = \int_0^\infty z^j f_Z(z)dz \tag{1f}$$

The term $M(\xi + \Delta) - M(\xi)$ represents the number of claims advised between $(\xi + \Delta)$ and ξ while the insurer pays the total claim $s(\xi + \Delta) - s(\xi)$ between the times $(\xi + \Delta)$ and ξ. The term $|u + \boldsymbol{C}\xi - s(\xi)|$ is the deficit at the time of ruin while $\lim_{\xi \to \xi^-} \left(u + \boldsymbol{C}\xi - s(\xi)\right)$ defines the surplus just before the ruin. The function

$$\left\{|u + \boldsymbol{C}\xi - S(\xi)| + \lim_{\xi \to \xi^-} \left(u + \boldsymbol{C}\xi - s(\xi)\right)\right\} \tag{1g}$$

accounts for the value of the insurance portfolio in ruin. The term $\{M(\xi); \xi \geq 0\}$ is a Poisson process with intensity rate $\lambda > 0$. The sequence $\{z_j\}_{j\geq 1}$ is independently and identically distributed with $F_X(z)$ and with $F_X(0) = 0$ where the mth moment $\mu_m = \boldsymbol{E}(Z^m)$. In the model, we assume that $\{z_j\}_{j\geq 1}$ and $\{M(\xi); \xi \geq 0\}$ are independent and again, it is also assumed that there exists some positive market price (loading factor) θ under the condition that $\theta = \boldsymbol{C}(\lambda\mu)^{-1} - 1 > 0$ is strictly positive. For computational conveniences, it is assumed that the aggregate claim process $\{S(\xi); \xi \geq 0\}$ is a compound Poisson process.

Let $F(z) = \boldsymbol{P}(Z_1 \leq z) = 1 - S(z)$ be the distribution function of individual claims. It is assumed that $F(z)$ has density function $f(z)$ such that $F(0) = 0$ such that claims amounts are positive. In view of (1), the Laplace transform of Z_1 is then defined as

$$\tilde{f}(s) = \int_0^\infty e^{-sz} f(z)\, dz \tag{1h}$$

Suppose μ_1 is the mean claim amount and θ is the premium loading factor such that the condition $\boldsymbol{C} = (1 + \theta)\lambda\mu_1$ holds. Define T_u to be the time of ruin from the initial surplus u such that $T_u = inf\{\tau > 0: U(\tau) < 0\}$ and $T_u \to \infty$ if $U(\tau) \geq 0;\ \forall\ \tau > 0$. The ultimate ruin probability from the initial surplus u is defined by $\phi(u)$. Consequently, using equation (1e), the probability of ruin of the initial capital $U(0) > 0$ for the infinite time horizon is given by

$$\phi(u) = \boldsymbol{P}(Ruin|u = U(0)) = \boldsymbol{P}(T_u < \infty) \tag{1i}$$

Again, using equation (1e) for the finite time horizon T,

$$ruin = \boldsymbol{P}\big(u + \boldsymbol{C}\xi < S(\xi)\big); \quad 0 < \xi < T \tag{1j}$$

and for any $u > 0$, the finite time ruin probability is

$$\phi(u, T) = \boldsymbol{P}(Ruin|u = U(0)) = \boldsymbol{P}(T_u < T) \tag{1k}$$

Suppose $X(\xi)$ is considered over a small increment of time $(0, h)$. For us to obtain an integral equation for non-ruin probability $\varphi(u)$, Grandell (1991) examined $U(\xi)$ over a sufficiently small time interval $(0, h)$ under the following conditions.

(i) There is a zero claim in $(0, h)$ and the probability of this event is then

$$\boldsymbol{P}(zero\ \ claim) = 1 - \theta h + o(h) \tag{2}$$

where $o(h)$ is a function that vanishes.

(ii) One claim occurs in the interval $(0, h)$, but the claim will not lead to ruin. The probability of one claim in $(0, h)$ is

$$\boldsymbol{P}(one\ claim) = \theta h + o(h) \tag{2a}$$

(iii) A claim occurs in the interval $(0, h)$ and the value indemnified causes ruin.

(iv) Minimum of two claims occur in $(0, h)$ and the probability of two or more claims in the interval $(0, h)$ is

$\boldsymbol{P}(two\ or\ more\ claims) = o(h)$ where the notation $o(h)$ signifies a function that tends to zero.

Based on the conditions **(i)** to **(iv)**, the non-ruin probability φ is defined as follows

$$\varphi(u) = \left(1 - \theta h + o(h)\right)\varphi(u + \alpha h) + \left(\theta h + o(h)\right)\int_0^{u+\alpha h} \varphi(u + \alpha h - z)dF(z) + \left(\theta h + o(h)\right) \times o + o(h) \tag{2b}$$

$$\varphi(u) = (1 - \theta h)\varphi(u + \alpha h) + \theta h \int_0^{u+\alpha h} \varphi(u + \alpha h - z)dF(z) + o(h) \tag{3}$$

Since φ is differentiable, we can expand $\varphi(u + \alpha h)$ using the Taylor's series expansion about u

$$\varphi(u + \alpha h) = \varphi(u) + \varphi^{(1)}(u)(u + \alpha h - u) + \varphi^{(2)}(u)\frac{(\alpha h)^2}{2!} + \varphi^{(3)}(u)\frac{(\alpha h)^3}{4!} + \cdots + \varphi^{(n)}(u)\frac{(\alpha h)^n}{n!} \tag{4}$$

Ignoring the fourth and higher term, we have

$$\varphi(u + \alpha h) = \varphi(u) + \varphi^{(1)}(u) \times (\alpha h) + \varphi^{(2)}(u) \times \frac{(\alpha h)^2}{2!} \tag{5}$$

$$\varphi(u + \alpha h) = \varphi(u) + \alpha h\varphi'(u) + o(h) \tag{6}$$

Since $f_X(z)dz = dF_X(z)$, equation (2b) can re-expressed as

$$\varphi(u) = (1 - \theta h)\varphi(u + \alpha h) + \theta h \int_0^{u+\alpha h} \varphi(u + \alpha h - z)f_X(z)dz + o(h) \tag{7}$$

$$\varphi(u) = (1 - \theta h)[\varphi(u) + \alpha h\varphi'(u) + o(h)] + \theta h \int_0^{u+\alpha h} \varphi(u + \alpha h - z)f_X(z)dz + o(h) \tag{8}$$

$$\varphi(u) = \varphi(u) + \alpha h\varphi'(u) + o(h) - \theta h\varphi(u) - \theta h\alpha h\varphi'(u) + o(h) + \theta h \int_0^{u+\alpha h} \varphi(u + \alpha h - z)f_X(z)dz + o(h) \tag{9}$$

$$\varphi(u) - \varphi(u) = \alpha h\varphi'(u) - \theta h\varphi(u) - \theta h\alpha h\varphi'(u) + \theta h \int_0^{u+\alpha h} \varphi(u + \alpha h - z)f_X(z)dz + o(h) \tag{10}$$

$$0 = \alpha h\varphi'(u) - \theta h\varphi(u) - \theta h\alpha h\varphi'(u) + \theta h \int_0^{u+\alpha h} \varphi(u + \alpha h - z)f_X(z)dz + o(h) \tag{11}$$

$$\alpha h\varphi'(u) = \theta h\varphi(u) + \theta h\alpha h\varphi'(u) - \theta h \int_0^{u+\alpha h} \varphi(u + \alpha h - z)f_X(z)dz + o(h) \tag{12}$$

$$\alpha h\varphi'(u) - \theta h\alpha h\varphi'(u) = \theta h\varphi(u) - \theta h \int_0^{u+\alpha h} \varphi(u + \alpha h - z)f_X(z)dz + o(h) \tag{13}$$

$$\alpha h[\varphi'(u) - \theta h\varphi'(u)] = \theta h\varphi(u) - \theta h \int_0^{u+\alpha h} \varphi(u + \alpha h - z)f_X(z)dz \tag{14}$$

Dividing (14) through by αh, we obtain

$$[\varphi'(u) - \theta h\varphi'(u)] = \frac{\theta}{\alpha}\varphi(u) - \frac{\theta}{\alpha}\int_0^{u+\alpha h} \varphi(u + \alpha h - z)f_X(z)dz \tag{15}$$

Since $h < \varepsilon$ for small number $\varepsilon > 0$, we take the limit in (15) as $h \to 0$

$$\lim_{h\to 0}[\varphi'(u) - \theta h\varphi'(u)] = \frac{\theta}{\alpha}\lim_{h\to 0}\varphi(u) - \frac{\theta}{\alpha}\lim_{h\to 0}\int_0^{u+\alpha h} \varphi(u + \alpha h - z)f_X(z)dz \tag{16}$$

$$\varphi'(u) = \frac{\theta}{\alpha}\varphi(u) - \frac{\theta}{\alpha}\int_0^{u} \varphi(u - z)f_X(z)dz \tag{17}$$

Integrating (17) within the interval $(0, \xi)$ with respect to u, we have

$$\int_0^{\xi} \varphi'(u)du = \int_0^{\xi}\left[\frac{\theta}{\alpha}\varphi(u) - \frac{\theta}{\alpha}\int_0^{u}\varphi(u - z)dF(z)\right]du \tag{18}$$

$$[\varphi(u)]_0^{\xi} = \int_0^{\xi}\frac{\theta}{\alpha}\varphi(u)du - \int_0^{\xi}\frac{\theta}{\alpha}\int_0^{u}\varphi(u - z)dF(z)\,du \tag{19}$$

Evaluating the limits in (18), we have

$$\varphi(\xi) - \varphi(0) = \frac{\theta}{\alpha}\int_0^{\xi}\varphi(u)du - \frac{\theta}{\alpha}\int_0^{\xi}\int_0^{u}\varphi(u - z)dF(z)\,du \tag{20}$$

$$\varphi(\xi) - \varphi(0) = \frac{\theta}{\alpha}\int_0^{\xi}\varphi(u)du + \frac{\theta}{\alpha}\int_0^{\xi}\int_0^{u}\varphi(u - z)d\left(1 - F(z)\right)d \tag{21}$$

Note that

$$-dF(Z) = d\left(1 - F(z)\right) \tag{22}$$

$$\varphi(\xi) - \varphi(0) = \frac{\theta}{\alpha}\int_0^{\xi}\varphi(u)du + \frac{\theta}{\alpha}\left\{\int_0^{\xi}\left(\left[\varphi(u - z)\left(1 - F(z)\right)\right]_0^{u} - \int_0^{u}\left(1 - F(z)\right)d\varphi(u - z)\right)du\right\} \tag{23}$$

$$\varphi(\xi)-\varphi(0)=\frac{\theta}{\alpha}\int_0^{\xi}\varphi(u)du+\frac{\theta}{\alpha}\int_0^{\xi}\{\varphi(0)(1-F(u))-\varphi(u)(1-F(0))\}+\frac{\theta}{\alpha}\int_0^u(1-F(z))d\quad(u-z)\,du \qquad (24)$$

$$\varphi(\xi)-\varphi(0)=\frac{\theta}{\alpha}\int_0^{\xi}\varphi(u)du+\frac{\theta}{\alpha}\varphi(0)\int_0^{\xi}(1-F(u))du-\frac{\theta}{\alpha}\int_0^{\xi}\varphi(u)du+\frac{\theta}{\alpha}\int_0^{\xi}\int_0^u(1-F(z))d\varphi(u-z)dz\,du \qquad (25)$$

$$\varphi(\xi)-\varphi(0)=\frac{\theta}{\alpha}\int_0^{\xi}\varphi(u)du+\frac{\theta}{\alpha}\varphi(0)\int_0^{\xi}(1-F(u))du-\frac{\theta}{\alpha}\int_0^{\xi}\varphi(u)du+\frac{\theta}{\alpha}\int_0^{\xi}\int_0^u(1-F(z))d\varphi(u-z)dz\,du \qquad (26)$$

$$\varphi(\xi)-\varphi(0)=\frac{\theta}{\alpha}\varphi(0)\int_0^{\xi}(1-F(u))du+\frac{\theta}{\alpha}\int_0^{\xi}\int_0^u(1-F(z))\varphi'(u-z)dz\,du \qquad (27)$$

$$\varphi(\xi)-\varphi(0)=\frac{\theta}{\alpha}\varphi(0)\int_0^{\xi}(1-F(u))d\quad+\frac{\theta}{\alpha}\int_0^{\xi}\int_0^u(1-F(z))\varphi'(u-z)dz\,du \qquad (28)$$

$$\varphi(\xi)-\varphi(0)=\frac{\theta}{\alpha}\int_0^{\xi}[1-F(z)]\,\varphi(t-z)dz \qquad (29)$$

$$\varphi(\xi)=\varphi(0)+\frac{\theta}{\alpha}\int_0^{\xi}[1-F(z)]\,\varphi(t-z)dz \qquad (30)$$

$$\varphi(u)=\varphi(0)+\frac{\theta}{\alpha}\int_0^{u}[1-F(z)]\,\varphi(u-z)dz \qquad (31)$$

Following Asmussen (1984); and Gray and Pitts (2012, pp. 287), we take the limit of both sides to enable us apply the monotone convergence theorem

$$\lim_{u\to\infty}\varphi(u)=\lim_{u\to\infty}\varphi(0)+\frac{\theta}{\alpha}\lim_{u\to\infty}\int_0^u[1-F(z)]\,\varphi(u-z)dz \qquad (32)$$

$$\varphi(\infty)=\varphi(0)+\frac{\theta}{\alpha}\int_0^{\infty}[1-F(z)]\,\varphi(\infty)dz \qquad (33)$$

$$\varphi(\infty)=\varphi(0)+\frac{\theta}{\alpha}\varphi(\infty)\int_0^{\infty}S(z)\,dz \qquad (34)$$

$$\varphi(\infty)=\varphi(0)+\left(\frac{\theta}{\alpha}\right)\times\varphi(\infty)\mu \qquad (36)$$

where

$$E(z)=\mu=\int_0^{\infty}S(z)\,dz \qquad (36a)$$

Following Asmusen & Albrecher (2010); and Asmusen & Albrecher (2007), we make the following assumption $\varphi(\infty)=1$ and then

$$\varphi(0)+\frac{\theta}{\alpha}\mu=1 \qquad (37)$$

$$\varphi(0)=1-\frac{\theta}{\alpha}\mu \qquad (38)$$

$$\frac{\theta}{\alpha}\mu=1-\varphi(0)=\gamma(0) \qquad (39)$$

The loading factor D is defined as follows

$$D=\alpha\times\varphi(0)=\alpha-\theta u \qquad (40)$$

Letting

$$\beta=\frac{\alpha-\theta}{\theta u}=\frac{\alpha}{\theta u}-1 \qquad (41)$$

then

$$(1+\beta)\times\theta u=\alpha \qquad (42)$$

we can take inverses

$$\frac{1}{(1+\beta)}=\frac{\theta u}{\alpha}=1-\varphi(0)=\gamma(0) \qquad (43)$$

So that

$$\varphi(0) = 1 - \frac{1}{1+\beta} \quad (44)$$

$$\varphi(0) = \frac{1+\beta-1}{1+\beta} \quad (45)$$

$$\varphi(0) = \frac{\beta}{1+\beta} \quad (46)$$

$$\varphi'(u) = \frac{\theta}{\alpha}\varphi(u) - \frac{\theta}{\alpha}\int_0^u \varphi(u-z)dF(z) \quad (47)$$

Letting

$$F(z) = 1 - e^{-\frac{z}{\mu}} \Rightarrow \mu dF(z) = e^{-\frac{z}{\mu}}dz \quad (48)$$

Using equation (47)

$$\varphi'(u) = \frac{\theta}{\alpha}\varphi(u) - \frac{\theta}{\alpha\mu}\int_0^u \varphi(u-z)e^{-\frac{z}{\mu}}dz \quad (49)$$

Letting

$$\varpi = u - z \Rightarrow d\varpi = -dz \quad (50)$$

$$\varphi'(u) = \frac{\theta}{\alpha}\varphi(u) - \frac{\theta}{\alpha\mu}\int_u^0 \varphi(\varpi)e^{-\frac{(u-\varpi)}{\mu}}(-1)d\varpi \quad (51)$$

$$\varphi'(u) = \frac{\theta}{\alpha}\varphi(u) - \frac{\theta}{\alpha\mu}\int_0^u \varphi(\varpi)e^{-\frac{(u-\varpi)}{\mu}}d\varpi \quad (52)$$

$$-\left(\varphi'(u) - \frac{\theta}{\alpha}\varphi(u)\right) = \frac{\theta}{\alpha\mu}\int_0^u \varphi(\varpi)e^{-\frac{(u-\varpi)}{\mu}}d\varpi \quad (53)$$

$$\varphi''(u) = \frac{\theta}{\alpha}\varphi'(u) - \frac{\theta}{\alpha\mu}\left(\varphi(u) - \int_0^u \varphi(\varpi)\frac{1}{\mu}e^{-\frac{(u-\varpi)}{\mu}}d\varpi\right) \quad (54)$$

$$\varphi''(u) = \frac{\theta}{\alpha}\varphi'(u) - \frac{\theta}{\alpha\mu}\varphi(u) - \frac{1}{\mu}\left(\varphi'(u) - \frac{\theta}{\alpha}\varphi(u)\right) \quad (55)$$

$$\varphi''(u) = \frac{\theta}{\alpha}\varphi'(u) - \frac{\theta}{\alpha\mu}\varphi(u) - \frac{1}{\mu}\varphi'(u) + \frac{1}{\mu}\frac{\theta}{\alpha}\varphi(u) \quad (56)$$

$$\varphi''(u) - \left(\frac{\theta}{\alpha} - \frac{1}{\mu}\right)\varphi'(u) = 0 \quad (57)$$

$$\varphi(u) = \omega_1\, exp\left(\frac{\theta}{\alpha} - \frac{1}{\mu}\right)u + \omega_2\, exp\left(-\left(\frac{\theta}{\alpha} - \frac{1}{\mu}\right)\right)u \quad (58)$$

where ω_1 and ω_1 are constants. By the boundary condition in (46)

$$\varphi(0) = \omega_1 + \omega_2 = \frac{\beta}{1+\beta} \quad (59)$$

Also, the monotone convergence theorem requires that $\varphi(\infty) = 1$

Consequently,

$$\varphi(\infty) = \omega_1\, exp\left(\frac{\theta}{\alpha} - \frac{1}{\mu}\right) \times (\infty) + \omega_2\, exp\left(-\left(\frac{\theta}{\alpha} - \frac{1}{\mu}\right)\right) \times (\infty) \quad (59a)$$

$$\varphi(\infty) = \omega_1 \times 1 + \omega_2 \times 0 = 1 \quad (59b)$$

$$\omega_1 = 1 \quad (59c)$$

Equation (59) becomes

$$1 + \omega_2 = \frac{\beta}{1+\beta} \quad (59d)$$

$$\omega_2 = 1 - \frac{\beta}{1+\beta} \quad (59e)$$

Similarly, equation (59a) becomes

$$\varphi(u) = exp\left(\frac{\theta}{\alpha} - \frac{1}{\mu}\right)u + \left(1 - \frac{\beta}{1+\beta}\right) exp\left(-\left(\frac{\theta}{\alpha} - \frac{1}{\mu}\right)\right)u \quad (59f)$$

Observe that

$$U(\xi) = u + \alpha\xi - \sum_{i=1}^{M(\xi)} z_i \quad (61)$$

Where $U(0) = u$is the initial capital, α is the premium rate per unit time, $\{M(\xi)\}$ is a homogeneous Poisson process, $z_1, z_2, \ldots$ are identically and independently distributed with $F(.)$ and $z_1, z_2, \ldots$ are independent of $\{M(\xi)\}$. An underwriter may dictate its own volume of business by adjusting the premium $\boldsymbol{C}$. As the premium intensity increases, the number of policies in its insurance portfolio reduces and such the claim intensities reduce. In order to define the amount of premium to advise, it is necessary to obtain a correct trajectory of the premium function $\pi(\xi)$ in equation (61). This is estimated in line with the order of magnitude of the process $s(\xi)$ which defines the aggregate losses of the surplus process$U(\xi) = u + \pi(\xi) - s(\xi)$. An adequate estimation of the trajectory of the premium function$\pi(\xi)$ is necessary because if the premium function is very steep, it is possible the insurer may not have a competitive advantage in the market. However, if $\pi(\xi)$ is chosen to be a slowly increasing function, the insurer could be ruined. It is, therefore, complex to obtain the distribution of $s(\xi)$ and, as such, becomes an intractable problem. For the insurers to avoid ruinous conditions, a reasonable premium estimation policy is formulated to cover eventual losses expressed in terms of $s(\xi)$.

$$\boldsymbol{E}(M(\xi)) = \lambda\xi = \boldsymbol{V}(M(\xi)) \tag{62}$$

where $M(\xi)$ is a homogeneous Poisson process with intensity λ However,

$$\boldsymbol{E}(S(\xi)) = \boldsymbol{E}\left(\sum_{i=1}^{M(\xi)} z_i\right) = \boldsymbol{E}\sum_{j=0}^{\infty}[(\sum_{i=1}^{j} z_i) \times 1_{\{M(\xi)=j\}}] \tag{63}$$

$$\boldsymbol{E}(S(\xi)) = \sum_{j=0}^{\infty}[\boldsymbol{E}(1_{\{M(\xi)=j\}}) \times \boldsymbol{E}(z_1 + z_2 + z_3 + \ldots + z_j)] \tag{64}$$

$$\boldsymbol{E}(S(\xi)) = \sum_{j=0}^{\infty} P(M(\xi) = j) \times j \times \boldsymbol{E}(z_1) \tag{65}$$

$$\boldsymbol{E}(S(\xi)) = \boldsymbol{E}(z_1)\sum_{j=0}^{\infty} jP(M(\xi) = j) \tag{66}$$

$$\boldsymbol{E}(S(\xi)) = \boldsymbol{E}(z_1)\boldsymbol{E}(M(\xi)) \tag{67}$$

$$\boldsymbol{E}(S(\xi)) = \lambda\xi \times \boldsymbol{E}(z_1) \tag{68}$$

Consequently,

$$\boldsymbol{V}(S(\xi)) = \boldsymbol{E}(S(\xi))^2 - [\boldsymbol{E}(S(\xi))]^2 \tag{69}$$

$$\boldsymbol{V}(S(\xi)) = \boldsymbol{E}(S(\xi))^2 - \lambda^2\xi^2 \times (\boldsymbol{E}(z_1))^2 \tag{70}$$

$$\boldsymbol{V}(S(\xi)) = \boldsymbol{V}(z_1)E(M(\xi)) + (\boldsymbol{E}(z_1))^2\boldsymbol{V}(M(\xi)) \tag{71}$$

$$\boldsymbol{V}(S(\xi)) = \lambda\xi \times \boldsymbol{V}(z_1) + \lambda\xi \times (\boldsymbol{E}(z_1))^2 \tag{72}$$

$$\lim_{\xi\to\infty}\left(\frac{\boldsymbol{V}(z_1)E(M(\xi))}{\xi}\right) = \lim_{\xi\to\infty}\left(\frac{\lambda\xi\times\boldsymbol{V}(z_1)}{\xi}\right) = \lambda \times \boldsymbol{V}(z_1) \tag{73}$$

The behavior of the claim process $s(\xi)$ as $\xi \to \infty$ defines the aggregate claim growing linearly with respect to time in the surplus process. The premium should increase linearly with its gradient greater than $\lambda\boldsymbol{E}(Z_1)$. This accounts for the reason why the premium function,

$$\pi(\xi) = \boldsymbol{C}\xi \tag{74}$$

Assumes a linear form

Gamma Claims Distribution

If the random loss variable$X$has a gamma distribution then its probability density function is defined by

$$f_X(x) = \frac{\lambda}{\Gamma(\alpha)}(\lambda x)^{\alpha-1}e^{-\lambda x} \tag{75}$$

and the mean loss is given by

$$\boldsymbol{E}(x) = \int_0^{\infty} xf_X(x)dx \tag{75a}$$

$$\boldsymbol{E}(x) = \frac{\alpha}{\lambda} \tag{76}$$

This is the mean of the gamma distribution with parameter $\lambda and \alpha$

The kthmoment is expressed as

$$\boldsymbol{E}(x^k) = \frac{(\alpha+k-1)(\alpha+k-2)(\alpha+k-3)\ldots\alpha}{\lambda^k} \tag{77}$$

$$\boldsymbol{V}(x) = \frac{\alpha(\alpha+1)}{\lambda^2} - \frac{\alpha^2}{\lambda^2} = \frac{\alpha^2+\alpha-\alpha^2}{\lambda^2} \tag{78}$$

$$\boldsymbol{V}(x) = \frac{\alpha}{\lambda^2} \tag{79}$$

$$M_X(t) = \boldsymbol{E}(e^{tx}) = \int_0^\infty e^{tx} f_X(x)dx \quad (80)$$

$$M_X(\xi) = \frac{\lambda^\alpha}{(\lambda-\xi)^\alpha} = \left(\frac{\lambda}{\lambda-\xi}\right)^\alpha \quad (81)$$

which is the moment generating function of the gamma distribution with parameter λ $and\ \alpha$.

III. Methodology

The Lundberg's coefficient is defined by

$$\phi_L(u) \leq \frac{e^{-Ru} M_X(R)\lambda}{R(1+\theta)\mu\lambda+\lambda} = e^{-Ru} \quad (81a)$$

while Tijim's model is given by

$$\phi_T(u) = \left(\frac{1}{1+\theta} - A\right) e^{-\frac{u}{\alpha}} + Ae^{-R} \quad (81b)$$

Data Presentation and Analysis

Motor insurance data in respect of 2022-2023 claims was sourced from a top Nigerian insurance firm and then cleaned before being analysed. The moments $\boldsymbol{E}(X^k);\ \ k = 1,2,3$ of the assumed gamma claims were computed using equation (77) and the adjustment coefficient $R$was hence computed using equation (138). Only the initial capital in the first column of tables 2and 3are the author's hypothetical data. The Tijims ruin was computed using (81b) while the Lundberg's upper bound was computed using (81a). The analysis was done on R software.

Table 1. The Adjustment Coefficient for Anonymous Insurance

R when $\theta = 0.1$	R when $\theta = 0.2$	R when $\theta = 0.3$
0.00000002252475	0.00000003985551	0.00000005448194

Source: Author's Computation

The result from table 1 showed that the adjustment coefficient increases as the safety loading increases.

Ruin Probability Using Lundberg's Equation $\phi(u) = e^{-Ru}$ based on the Values of the Adjustment Coefficient R for the initial capital u when the values of safety loading θ are 0.1, 0.2 and 0.3 respectively.

Table 2. Lundberg's Ruin Probability for Anonymous Insurance

Initial Capital	$\phi(u)$ when $\theta = 0.1$	$\phi(u)$ when $\theta = 0.2$	$\phi(u)$ when $\theta = 0.3$
1000000	1.018987	0.952975	0.853312
1100000	1.016694	0.949184	0.848676
1200000	1.014407	0.945409	0.844065
1300000	1.012124	0.941648	0.839479
1400000	1.009847	0.937903	0.834918
1500000	1.007575	0.934172	0.830381
1600000	1.005308	0.930456	0.825869
1700000	1.003046	0.926755	0.821382
1800000	1.000789	0.923069	0.816919
1900000	0.998537	0.919397	0.812481

2000000	0.996291	0.915740	0.808066
2100000	0.994049	0.912098	0.803676
2200000	0.991813	0.908470	0.799309
2300000	0.989581	0.904856	0.794966
2400000	0.987355	0.901257	0.790647
2500000	0.985133	0.897672	0.786351
2600000	0.982917	0.894102	0.782078
2700000	0.980705	0.890545	0.777829
2800000	0.978499	0.887003	0.773603
2900000	0.976297	0.883475	0.769399
3000000	0.974101	0.879961	0.765219
3100000	0.971909	0.876461	0.761061
3200000	0.969722	0.872974	0.756926
3300000	0.967540	0.869502	0.752813
3400000	0.965363	0.866043	0.748723
3500000	0.963191	0.862599	0.744655
3600000	0.961024	0.859168	0.740609
3700000	0.958862	0.855750	0.736585
3800000	0.956705	0.852346	0.732583
3900000	0.954552	0.848956	0.728602
4000000	0.952404	0.845579	0.724644
4100000	0.950262	0.842216	0.720706
4200000	0.948124	0.838866	0.716790
4300000	0.945990	0.835529	0.712896
4400000	0.943862	0.832206	0.709022
4500000	0.941738	0.828895	0.705170
4600000	0.939619	0.825598	0.701339
4700000	0.937505	0.822314	0.697528
4800000	0.935396	0.819044	0.693738
4900000	0.933291	0.815786	0.689969
5000000	0.931192	0.812541	0.686220
5100000	0.929096	0.809309	0.682491
5200000	0.927006	0.806090	0.678783
5300000	0.924920	0.802883	0.675095
5400000	0.922839	0.799690	0.671427
5500000	0.920763	0.796509	0.667779
5600000	0.918691	0.793341	0.664151
5700000	0.916624	0.790185	0.660542
5800000	0.914562	0.787042	0.656953
5900000	0.912504	0.783912	0.653383
6000000	0.910451	0.780794	0.649833
6100000	0.908403	0.777688	0.646303
6200000	0.906359	0.774594	0.642791
6300000	0.904320	0.771513	0.639298
6400000	0.902285	0.768445	0.635825
6500000	0.900255	0.765388	0.632370

6600000	0.898229	0.762344	0.628934
6700000	0.896208	0.759311	0.625517
6800000	0.894192	0.756291	0.622118
6900000	0.892180	0.753283	0.618738
7000000	0.890173	0.750287	0.615376
7100000	0.888170	0.747302	0.612033
7200000	0.886172	0.744330	0.608707
7300000	0.884178	0.741369	0.605400
7400000	0.882188	0.738420	0.602111
7500000	0.880203	0.735483	0.598839
7600000	0.878223	0.732558	0.595585
7700000	0.876247	0.729644	0.592349
7800000	0.874276	0.726741	0.589131
7900000	0.872309	0.723851	0.585930
8000000	0.870346	0.720972	0.582746
8100000	0.868388	0.718104	0.579580
8200000	0.866434	0.715247	0.576431
8300000	0.864484	0.712402	0.573299
8400000	0.862539	0.709569	0.570184
8500000	0.860599	0.706746	0.567086
8600000	0.858662	0.703935	0.564005
8700000	0.856731	0.701135	0.560940
8800000	0.854803	0.698346	0.557893
8900000	0.852880	0.695569	0.554861
9000000	0.850961	0.692802	0.551847
9100000	0.849046	0.690046	0.548848
9200000	0.847136	0.687302	0.545866
9300000	0.845230	0.684568	0.542900
9400000	0.843328	0.681845	0.539950
9500000	0.841431	0.679133	0.537017
9600000	0.839538	0.676431	0.534099
9700000	0.837649	0.673741	0.531197
9800000	0.835764	0.671061	0.528311
9900000	0.833884	0.668392	0.525440
10000000	0.832007	0.665733	0.522585

Source: Author's own source

Ruin Probability Using Tjiim's Approximation Based on the Values of the Adjustment Coefficient R.

Table 3. Tijim's Ruin Probability for Anonymous Insurance

INITIAL CAPITAL (u)	$\phi_T(u)$ when $\theta = 0.1$	$\phi_T(u)$ when $\theta = 0.2$	$\phi_T(u)$ when $\theta = 0.3$
1000000	0.887751	0.798430	0.725482

1100000	0.885644	0.795019	0.721242
1200000	0.883542	0.791622	0.717026
1300000	0.881445	0.788239	0.712833
1400000	0.879353	0.784870	0.708665
1500000	0.877266	0.781515	0.704519
1600000	0.875183	0.778174	0.700398
1700000	0.873106	0.774847	0.696300
1800000	0.871033	0.771533	0.692224
1900000	0.868965	0.768233	0.688172
2000000	0.866902	0.764948	0.684143
2100000	0.864843	0.761675	0.680137
2200000	0.862790	0.758416	0.676153
2300000	0.860741	0.755171	0.672192
2400000	0.858697	0.751939	0.668253
2500000	0.856658	0.748721	0.664337
2600000	0.854623	0.745516	0.660442
2700000	0.852593	0.742324	0.656570
2800000	0.850568	0.739146	0.652720
2900000	0.848548	0.735981	0.648891
3000000	0.846532	0.732829	0.645084
3100000	0.844521	0.729690	0.641299
3200000	0.842515	0.726564	0.637535
3300000	0.840513	0.723451	0.633792
3400000	0.838516	0.720351	0.630071
3500000	0.836524	0.717263	0.626371
3600000	0.834536	0.714189	0.622691
3700000	0.832553	0.711128	0.619033
3800000	0.830574	0.708079	0.615395
3900000	0.828600	0.705042	0.611778
4000000	0.826631	0.702019	0.608181
4100000	0.824666	0.699008	0.604605
4200000	0.822706	0.696009	0.601049
4300000	0.820750	0.693023	0.597513
4400000	0.818799	0.690050	0.593997
4500000	0.816853	0.687088	0.590502
4600000	0.814911	0.684139	0.587026
4700000	0.812973	0.681203	0.583569
4800000	0.811040	0.678278	0.580133
4900000	0.809112	0.675366	0.576715
5000000	0.807188	0.672466	0.573318
5100000	0.805268	0.669578	0.569939
5200000	0.803353	0.666701	0.566580
5300000	0.801443	0.663837	0.563239
5400000	0.799537	0.660985	0.559918
5500000	0.797635	0.658145	0.556616
5600000	0.795738	0.655316	0.553332

5700000	0.793845	0.652499	0.550067
5800000	0.791956	0.649694	0.546820
5900000	0.790072	0.646901	0.543592
6000000	0.788192	0.644119	0.540383
6100000	0.786317	0.641349	0.537191
6200000	0.784446	0.638590	0.534018
6300000	0.782579	0.635843	0.530862
6400000	0.780717	0.633107	0.527725
6500000	0.778859	0.630383	0.524605
6600000	0.777006	0.627670	0.521503
6700000	0.775156	0.624968	0.518419
6800000	0.773311	0.622277	0.515353
6900000	0.771471	0.619598	0.512303
7000000	0.769634	0.616930	0.509271
7100000	0.767802	0.614273	0.506257
7200000	0.765974	0.611627	0.503259
7300000	0.764151	0.608992	0.500279
7400000	0.762331	0.606368	0.497315
7500000	0.760516	0.603755	0.494369
7600000	0.758705	0.601153	0.491439
7700000	0.756898	0.598562	0.488525
7800000	0.755096	0.595981	0.485629
7900000	0.753298	0.593411	0.482749
8000000	0.751504	0.590852	0.479885
8100000	0.749714	0.588304	0.477038
8200000	0.747928	0.585766	0.474206
8300000	0.746146	0.583239	0.471391
8400000	0.744369	0.580722	0.468592
8500000	0.742595	0.578216	0.465809
8600000	0.740826	0.575721	0.463042
8700000	0.739061	0.573235	0.460290
8800000	0.737300	0.570760	0.457554
8900000	0.735543	0.568296	0.454834
9000000	0.733791	0.565841	0.452130
9100000	0.732042	0.563397	0.449440
9200000	0.730297	0.560964	0.446766
9300000	0.728557	0.558540	0.444108
9400000	0.726820	0.556126	0.441464
500000	0.725088	0.553723	0.438836
9600000	0.723359	0.551329	0.436222
9700000	0.721635	0.548946	0.433624
9800000	0.719915	0.546572	0.431040
9900000	0.718198	0.544208	0.428471
10000000	0.716486	0.541855	0.425916

Source: Author's own source

IV. Findings and Discussion

Under the safety loadings $0.1, 0.2\ and\ 0.3$ shown in Table 1, the probability that the ruin probability at $u$falls below u, that is $\boldsymbol{P}(U(\xi) < u) = \frac{1}{1+\theta}$ are $\{0.990099; 0.980392; 0.970874\}$.The probabilities progressively decline as the safety loading increases, while the adjustment coefficient increases as the safety loading increases. However, the implication is that the riskier the insurance portfolio becomes as the value of Rbecomes smaller. The result from table $2-3$ showed that as the level of Lundberg's ruin probabilities decreases, the size of the initial capital increases establishing an inverse linear relationship and consequently the probability of ruin will tend to zero when the initial capital $u$is large with the exponential bound expressed in equation (104). The result of Lundberg's model is quite informative than the average behavior of the insurance portfolio as expressed in the net profit condition since it is always assumed that the initial capital is given. In Table2, the ruin probability is less than or equal to 1 when $\theta = 0.1$ while at $\theta = 0.2$ and $\theta = 0.3$, the ruin probability is strictly less than 1. Therefore, the insurance firm should avoid initial reserve below1,800,000.00. From Tables 2 and 3, at any level of the initial capital u both models seem not to converge very fast to zero. Within the interval$1000000 \leq u \leq 1800000$, the ruin probability is trivially 1 confirming that the net profit condition $\lambda E(Z) - \boldsymbol{C} < 0$ assumed in the surplus process $U(\xi) = u + \boldsymbol{C}\xi - \sum_{i=1}^{M(\xi)} z_i$ is violated. Furthermore, because the net profit condition implicitly implies that the claims incurred on the average would not occur frequently than premiums received, any contravention of this condition would definitely ensure ruinous conditions especially where both the claim inter-occurrence time and random losses become degenerate.

Therefore, the accumulated net loss at time ξ associated with the surplus process $U(\xi)$ can be expressed as

$$h(\xi) = \max_{\xi}[s(\xi) - \boldsymbol{C}\xi] \qquad (81c)$$

Since $u \geq 0$ then

$$1 - \phi(u) = 1 - \boldsymbol{P}(U(\xi) < 0; \xi > 0) \qquad (81d)$$

$$1 - \boldsymbol{P}(U(\xi) < 0; \xi > 0) = \boldsymbol{P}(U(\xi) \geq 0; \xi > 0) \qquad (81e)$$

$$1 - \boldsymbol{P}(U(\xi) < 0; \xi > 0) = \boldsymbol{P}(u + \boldsymbol{C}\xi - s(\xi) \geq 0; \xi > 0) \qquad (81f)$$

$$1 - \boldsymbol{P}(U(\xi) < 0; \xi > 0) = \boldsymbol{P}(u \geq s(\xi) - \boldsymbol{C}\xi; \xi > 0) \qquad (81g)$$

$$\boldsymbol{P}(h \leq u) = 1 - \boldsymbol{P}(U(\xi) < 0; \xi > 0) \qquad (81h)$$

Consequently, the expected claim value per unit time will be less than the insurer's premium per unit time. Although in practice the ruin probability$\phi(u) = \boldsymbol{P}(min\, U\,(\xi) < 0; \xi > 0)$ is formulated in comparing both models, it does not really imply that the insurer has become insolvent when the insurer's surplus fall below a defined benchmark for the first time. What this connotes is that the initial reserve could describe the capital which the underwriter would likely risk and such that if ruin eventually occurs at this point, the insurer should reasonably take proactive actions to ensure that underwriting business is profitable. Empirical evidence from table $II - III$ reveals that the Tijim's estimation to ruin probability drops below the Lundberg's approximation at the same level of initial capital and safety loading. We observe in equation (110), that the Tijim's approximation is applicable only when $R > \frac{1}{\alpha}$ and for large values of u, the Tijim's approximation would agree with Lundberg's estimation. In particular the Tijim's ruin value at infinity $\phi_T(\infty) = 0$ while if $u = 0$, $\phi_T(0) = \frac{1}{1+\theta}$. The justification of the Tijim's estimation is to guarantee an improvement over Lundberg's model and hence Tijim's ruin model may be most appropriate for the insurance firm depending on its risk appetite.

Based on the discussions so far, the results obtained could be used by Nigerian insurance regulators to formulate policy framework to regulate insurance industry so as to forestall gross consequences of ruinous conditions. This may also assist the consulting actuaries to advise the regulators on policy guidelines which improve the required minimum capital regime.

V. Conclusion

The random behavior of an insurer's underwriting business and its duty to oblige future claims have attracted the keen interest of insurance regulators. As an immediate consequence, the risk-based capital to hold has become a hydra-headed problem for insurance firms. This has orchestrated multi-dimensional waves of discourse associated with the functional relationship within the following analytics (1) initial level of reserve (2) the probability of ruin and (3) the behavior of the underlying business risk. Therefore, to maintain a good financial health of the insurance industry, the Nigerian insurance regulators need to enforce series of proactive measures regularly including capital requirements to guide against insolvency. In this paper, a closed form explicit solution of the adjustment co-efficient has been developed to represent a unique solution of the Lundberg's equation. Since the closed form expression would not require an initial guess common in Newton-Raphson numerical technique and can be differentiated to conduct stress analysis, it definitely has an edge over other numerically computational techniques for solving the Lundberg's equation. From the results presented, the computed ruin probabilities in the continuous-time classical surplus process is assumed to exist if and only if the insurance claim distribution is closely approaching an gamma distribution. In this work, we have compared two explicit approximate ruin probabilities for a claim amount distribution. The numerical computations reveal that almost all the estimated ruin probabilities by Tijim's estimation satisfactorily fall below the upper bound approximation of ruin probability and consequently, the technique resulted in better performance than the Lundberg's estimation. Therefore, Tijim's approximation could be useful in obtaining the initial capital for managing the ruin probability of insurance firms over infinite time since it has not exceeded the upper bound of ruin probability.

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Appendix I

The Lundberg's Coefficient

The Lundberg's approximation was developed to assess the optimal level of the ruin probabilities of an insurer's portfolio in predicting its insolvency. The essence of the model is to modify the estimation problem of ruin's upper bound probability to solving the adjustment co-efficient equation.

At an arbitrary time $\xi > 0$ and with the adjustment coefficient $R > 0$

$$\boldsymbol{E}\left(e^{-RU(\xi)}\right) = \boldsymbol{E}\left(e^{-RU(\xi)}|\tau < \xi\right) \times \boldsymbol{P}(\tau < \xi) + \boldsymbol{E}\left(e^{-RU(\xi)}|\tau \geq \xi\right) \times \boldsymbol{P}(\tau \geq \xi) \quad (82)$$

Recall that

$$U(\xi) = u + \boldsymbol{C}\xi - S(\xi) \quad (83)$$

Therefore

$$\boldsymbol{E}\left(e^{-RU(\xi)}\right) = \boldsymbol{E}\left(e^{-R[u+\boldsymbol{C}\xi-S(\xi)]}\right) \quad (84)$$

$$\boldsymbol{E}\left(e^{-RU(\xi)}\right) = \boldsymbol{E}\left(e^{[-Ru-R\boldsymbol{C}\xi+RS(\xi)]}\right) \quad (85)$$

$$\boldsymbol{E}\left(e^{-R\ \ (\xi)}\right) = \boldsymbol{E}\left(e^{[-Ru-R\boldsymbol{C}\xi]}\right) \times \boldsymbol{E}\left(e^{[RS(\xi)]}\right) \quad (86)$$

But

$$\boldsymbol{E}\left(e^{[RS(\xi)]}\right) = e^{\lambda\xi\times[M_X(r)-1]} \quad (87)$$

Therefore

$$\boldsymbol{E}\left(e^{-RU(\xi)}\right) = e^{[-Ru-R\boldsymbol{C}\xi]}e^{\lambda\xi\times[M_X(R)-1]} \quad (88)$$

$$\boldsymbol{E}\left(e^{-RU(\xi)}\right) = e^{[-Ru-R\boldsymbol{C}\xi]}e^{\lambda\xi\times[M_X(R)-1]} \quad (89)$$

For $\xi > \tau,\ U(\xi)$ can be expressed as

$$U(\xi) = u + \boldsymbol{C}\xi - s(\xi) \quad (90)$$

$$U(\tau) = u + \boldsymbol{C}\tau - s(\tau) \quad (91)$$

$$U(\xi) - U(\tau) = u + \boldsymbol{C}\xi - s(\xi) - \{u + \boldsymbol{C}\tau - s(\tau)\} \quad (92)$$

$$U(\xi) - U(\tau) = u + \boldsymbol{C}\xi - s(\xi) - u - \boldsymbol{C}\tau + s(\tau) \quad (93)$$

$$U(\xi) - U(\tau) = \boldsymbol{C}\xi - \boldsymbol{C}\tau + s(\tau) - s(\xi) \quad (94)$$

$$U(\xi) = U(\tau) + \boldsymbol{C}(\xi - \tau) - [S(\xi) - S(\tau)] \quad (95)$$

Since $s(\xi)$ is a Poisson process with independent increments, $U(\tau)$ also assumes a Poisson process with intensity $\lambda(\xi - \tau)$ and it is also independent of $\left(S(\xi) - S(\tau)\right)$

$$\boldsymbol{E}\left(e^{-RU(\xi)}|\tau < \xi\right) = \boldsymbol{E}\left(e^{-R[U(\tau)+\boldsymbol{C}(\xi-\tau)-(S(\xi)-S(\tau))]}|\tau < \xi\right) \quad (96)$$

$$\boldsymbol{E}\left(e^{-RU(\xi)}|\tau < \xi\right) = \boldsymbol{E}\left(e^{[-RU(\tau)-R\boldsymbol{C}(\xi-\tau)+R(S(\xi)+RS(\tau))]}|\tau < \xi\right) \quad (97)$$

$$\boldsymbol{E}\left(e^{-RU(\xi)}|\tau < \xi\right) = \boldsymbol{E}\left(e^{[-RU(\tau)-R\boldsymbol{C}(\xi-\tau)+\lambda(\xi-\tau)\times\{M_X(R)-1\}]}|\tau < \xi\right) \quad (98)$$

Equations (89) and (98) can both be re-expressed to obtain equation (82)

$$e^{-RU(\xi)} = \boldsymbol{E}\left(e^{-RU(\tau)}|\tau < \xi\right) \times \boldsymbol{P}(\tau < \xi) + \boldsymbol{E}\left(e^{-RU(\xi)}|\tau \geq \xi\right) \times \boldsymbol{P}(\tau \geq \xi) \quad (99)$$

$$\lim_{\xi\to\infty} e^{-RU(\xi)} = \lim_{\xi\to\infty}\boldsymbol{E}\left(e^{-RU(\tau)}|\tau < \xi\right) \times \boldsymbol{P}(\tau < \xi) + \lim_{\xi\to\infty}\boldsymbol{E}\left(e^{-RU(\xi)}|\tau \geq \xi\right) \times \boldsymbol{P}(\tau \geq \xi) \quad (100)$$

But by the observations in Bowers, Gerber, Hickman, Jones, Nesbit. (1997, chapter 13)

$$\lim_{\xi\to\infty}\boldsymbol{E}\left(e^{-RU(\xi)}|\tau \geq \xi\right) \times \boldsymbol{P}(\tau \geq \xi) = 0 \quad (101)$$

$$\lim_{\xi\to\infty} e^{-RU(\xi)} = \lim_{\xi\to\infty}\boldsymbol{E}\left(e^{-RU(\tau)}|\tau < \xi\right) \times \boldsymbol{P}(\tau < \xi) + 0 \quad (102)$$

The first time that the surplus becomes negative is the time of ruin $\tau = inf\{\xi : U(\xi) < 0\}$ with the assumption that while$\tau = \infty$ if $U(\xi) \geq 0$ for $\xi > 0$ so that ruin probability is given by $\phi(u) = \boldsymbol{P}(\tau < \infty)$. This accounts for the reason why we take the limit as ξ approaches infinity. In practice, this is the first time that the underwriter will be declared insolvent. Among the reserve requirements constituting an insurance balance sheet, there are specified technical provisions required by computing the expected value of the aggregate loss and consequently, it becomes clear that the balance sheet has major mathematical and actuarial implications. When the technical provisions are charged in conjuction with the safety loading, then the insurer has the prudence margin for technical provisions.

$$\lim_{\xi\to\infty} e^{-RU(\xi)} = \boldsymbol{E}\left(e^{-RU(\tau)}|\tau<\infty\right)\times\phi(u) \quad (103)$$

$$\frac{e^{-RU(\xi)}}{\boldsymbol{E}\left(e^{-RU(\tau)}|\tau<\infty\right)} = \phi(u) \quad (104)$$

But

$$\phi(u) = \boldsymbol{P}\left(\inf_{\xi>0}\{u+\boldsymbol{C}\xi - s(\xi)\} < 0\right) \quad (105)$$

$$\phi(u) = \boldsymbol{P}\left(\sup_{\xi>0}\{s(\xi)-\boldsymbol{C}\xi\} > u\right) \quad (106)$$

$$\phi(u) = \boldsymbol{P}\left(\sup_{\xi>0}\{u-U(\xi)\} > u\right) \quad (107)$$

$$\frac{e^{-R\ \ (\xi)}}{\boldsymbol{E}\left(e^{-RU(\tau)}|\tau<\infty\right)} = \boldsymbol{P}\left(\sup_{\xi>0}\{u-U(\xi)\} > u\right) \quad (108)$$

As $\theta\to 0$ then $R\to 0$ implying that ruin is approaching and $\boldsymbol{C}$ assumes the expected claim amount. Consequently, ruin will occur if the expected claim value is charged as underwriting premium. Again when $\tau<\infty$ then $U(\tau)<0$ and consequently $\boldsymbol{E}\left(e^{-R\ \ (\tau)}|\tau<\infty\right)>1$. Equation (108) is equivalent to

$$\phi_L(u) \le \frac{e^{-Ru}M_X(R)\lambda}{R(1+\theta)\mu\lambda+\lambda} = e^{-R} \quad (109)$$

Appendix II

Exponential co-efficient of Tijim's Approximation

Theorem

Given the Tijim's Approximation

$$\phi_T(u) = \{[\boldsymbol{P}(U(\xi)<u\wedge(u-z)-\delta z < U(\xi)<u-z)]-A\}e^{-\frac{u}{\alpha}} + (Ae^{-Ru}) \quad (110)$$

and $\mu_n = \boldsymbol{E}(X^n)$where $n\in \boldsymbol{z}^+$; then

$$\alpha = \frac{1}{([\boldsymbol{P}(U(\xi)<u)-A]-A)}\left[\frac{\sqrt{(3\mu_2)^2+48\theta\mu_1\mu_3}-3\mu_2}{4\mu_3} - \frac{A}{R}\right] \quad (111)$$

Proof

The adjustment co-efficient needs be estimated first

Observe that

$$u-u(\xi) = s(\xi)-\boldsymbol{C}\xi \quad (112)$$

The net claim payment within the interval $(0,\xi)$ is defined by

$$H(\xi) = s(\xi)-\boldsymbol{C}\xi \quad (113)$$

The moment generating function of $H(\xi)$ is computed as follows

$$M_{H(\xi)}(R) = \boldsymbol{E}\left[e^{R[s(\xi)-\boldsymbol{C}\xi]}\right] \quad (114)$$

$$M_{H(\xi)}(R) = \boldsymbol{E}\left[e^{R\times s(\xi)}e^{-\boldsymbol{C}R\xi}\right] = \boldsymbol{E}\left[e^{R\times s(\xi)}e^{-\boldsymbol{C}R\xi}\right] \quad (115)$$

$$M_{H(\xi)}(R) = e^{-\boldsymbol{C}R\xi}\boldsymbol{E}\left[e^{R\times s(\xi)}\right] = e^{-\boldsymbol{C}R\xi}M_{s(\xi)}(R) \quad (116)$$

$$M_{H(\xi)}(R) = e^{-\boldsymbol{C}R\xi}e^{\lambda\xi[M_X(R)-1]} \quad (117)$$

We assume that

$$M_{H(t)}(R) = e^0 \quad (118)$$

$$e^{-\boldsymbol{C}R\xi}e^{-\lambda\xi[M_X(R)-1]} = e^{-\boldsymbol{C}R\xi+\quad[M_X(R)-1]} = e^0 \quad (119)$$

$$log_e\left(e^{-\boldsymbol{C}R\xi\quad[M_X(R)-1]}\right) = log_e\, e^0 \quad (120)$$

Therefore

$$-\boldsymbol{C}R\xi + \lambda\xi[M_X(R)-1] = 0 \quad (121)$$

$$M_X(R) - 1 = \frac{\boldsymbol{C}R}{\lambda} \quad (122)$$

$$\lambda + \boldsymbol{C}R = \lambda M_X(R) \quad (123)$$

$$\lambda + \boldsymbol{C}R = \lambda \int_0^\infty e^{Rx} f_X(x)dx \quad (124)$$

$$\lambda + \boldsymbol{C}R = \lambda \int_0^\infty \left(1 + Rx + \frac{R^2x^2}{2} + \frac{R^3x^3}{3} + \frac{R^4x^4}{4} + \cdots\right) f_X(x)dx \quad (125)$$

Ignoring the fifth and higher terms, we have

$$\lambda + \boldsymbol{C}R \approx \lambda \int_0^\infty \left(1 + Rx + \frac{R^2x^2}{2} + \frac{R^3x^3}{3}\right) f_X(x)dx \quad (126)$$

$$\lambda + \boldsymbol{C}R = \lambda\begin{Bmatrix}\int_0^\infty f_X(x)\,dx + R\int_0^\infty x f_X(x)\,dx \\ + \frac{R^2}{2}\int_0^\infty x^2 f_X(x)\,dx + \frac{R^3}{3}\int_0^\infty x^3 f_X(x)\,dx\end{Bmatrix} \quad (127)$$

By the expected value principle,

$$\lambda + \boldsymbol{C}R = \lambda\left[1 + R \times \boldsymbol{E}(X) + \frac{R^2}{2}\boldsymbol{E}(X^2) + \frac{R^3}{3}\boldsymbol{E}(X^3)\right] \quad (128)$$

$$\lambda + \boldsymbol{C}R = \left[\lambda + \lambda R\mu_1 + \frac{\lambda R^2}{2}\mu_2 + \frac{\lambda R^3}{3}\mu_3\right] \quad (129)$$

$$\boldsymbol{C}R = \lambda R\mu_1 + \frac{\lambda R^2}{2}\mu_2 + \frac{\lambda R^3}{3}\mu_3 \quad (130)$$

$$\boldsymbol{C} = \lambda\mu_1 + \frac{\lambda R}{2}\mu_2 + \frac{\lambda R^2}{3}\mu_3 \quad (131)$$

But

$$\boldsymbol{C} = (1+\theta)\lambda\mu_1 \quad (132)$$

$$(1+\theta)\lambda\mu_1 = \frac{\lambda R^2}{3}\mu_3 + \frac{\lambda R}{2}\mu_2 + \lambda\mu_1 \quad (133)$$

$$2\lambda\mu_3R^2 + 3\lambda\mu_2R + 6\lambda\mu_1 - 6(1+\theta)\lambda\mu_1 = 0 \quad (134)$$

$$2\lambda\mu_3R^2 + 3\lambda\mu_2R + 6\lambda\mu_1 - 6\lambda\mu_1 - 6\lambda\theta\mu_1 = 0 \quad (135)$$

$$2\lambda\mu_3R^2 + 3\lambda\mu_2R - 6\lambda\theta\ \ _1 = 0 \quad (136)$$

$$R = \frac{-3\lambda\mu_2 \pm \sqrt{(3\lambda\mu_2)^2 + 48\lambda^2\theta\mu_1\mu_3}}{4\lambda\mu_3} \quad (137)$$

$$R = \frac{\sqrt{(3\mu_2)^2 + 48\theta\mu_1\mu_3} - 3\mu_2}{4\mu_3} \quad (138)$$

In order to estimate αin (110), we integrate as follows

$$\int_0^\infty \phi_T(u)du = [\boldsymbol{P}(U(\xi) < u) - A]\int_0^\infty e^{-\frac{u}{\alpha}}du + A\int_0^\infty e^{-Ru}\,du \quad (139)$$

Observe that for $z > 0$

$$[\boldsymbol{P}(U(\xi) < u \wedge (u-z) - \delta z < U(\xi) < u - z)] = \frac{\lambda}{\boldsymbol{C}}S_X(z)\delta z \quad (140)$$

and

$$\boldsymbol{P}(U(\xi) < u) = \int_0^\infty \frac{\lambda}{\boldsymbol{C}}S_X(z)dz = \frac{\lambda}{\boldsymbol{C}}\int_0^\infty S_X(z)dz \quad (141)$$

$$\mu_1 = \int_0^\infty S_X(z)dz \quad (142)$$

$$\boldsymbol{P}(U(\xi) < u) = \frac{\lambda}{\boldsymbol{C}}\boldsymbol{E}(X) = \frac{\lambda\mu_1}{\lambda(1+\theta)\mu_1} = \frac{1}{(1+\theta)} \quad (143)$$

$$\int_0^\infty \phi_T(u)du = \left(\frac{1}{1+\theta} - A\right)\left[-\alpha e^{-\frac{u}{\alpha}}\right]_0^\infty - \left[\frac{A}{R}e^{-Ru}\right]_0^\infty \quad (144)$$

$$\int_0^\infty \phi_T(u)du = \left(\frac{1}{1+\theta} - A\right)\alpha + \frac{A}{R} \qquad (145)$$

Using the identity

$$\int_0^\infty \phi_T(u)du \equiv \frac{1}{R} \qquad (146)$$

we obtain,

$$\left(\frac{1}{1+\theta} - A\right)\alpha + \frac{A}{R} = \frac{1}{R} \qquad (147)$$

$$\left(\frac{1}{1+\theta} - A\right)\alpha = \frac{1}{R} - \frac{A}{R} = \frac{1-A}{R} \qquad (148)$$

$$\left(\frac{1}{1+\theta} - A\right)\alpha = \frac{1}{R} - \frac{A}{R} = (1 - A)\left(\frac{1}{\frac{\sqrt{(3\mu_2)^2+48\theta\mu_1\mu_3}-3\mu_2}{4\mu_3}}\right) \qquad (149)$$

$$\alpha = \frac{(1-A)}{\left(\frac{1}{1+\theta}-A\right)}\left(\frac{4\mu_3}{\sqrt{(3\mu_2)^2+48\theta\mu_1\mu_3}-3\mu_2}\right) \qquad (150)$$

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where θ =Loading and $\mu = \boldsymbol{E}(X)$

$$\frac{\lambda\mu}{\boldsymbol{C}} = \frac{1}{(1+\theta)} \qquad (151)$$

and

$$A = \frac{\mu\theta}{M_X'(R)-\mu(1+\theta)} \qquad (152)$$

where A is from the Cramer's approximation.