Thetanulls of cyclic curves of genus 4

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Let $\aleph$ be an irreducible smooth projective cyclic curve of genus $g > 1$ defined over the complex field $\mathbb{C}$. These are by definition compact Riemann surfaces of genus $g > 1$ (unless we allow singular points) admitting an automorphism σ such that $\aleph/\sigma \cong \mathbb{P}^1$ and σ generates a normal subgroup of the automorphism group G of $\aleph$. When the curve is hyperelliptic, then the curve has extra automorphisms, in particular σ is not the hyperelliptic involution. The condition implies to having an equation $y^n = f(x)$ for the curve, where x is an affine coordinate on $\mathbb{P}^1$ and σ has order n. The branch points of $\pi : \aleph \to \mathbb{P}^1$, together with the signature of the cover provide algebraic coordinates for the curve in moduli. Choosing a symplectic homology basis $\{A_1, \cdots, A_g, B_1, \cdots B_g\}$ for a given curve $\aleph$ of genus $g \geq 2$ such that the intersection products $A_i.A_j = B_i.B_j = 0$ and $A_i.B_j = \delta_{ij}$ where δ_{ij} is the Kronecker delta and a basis $\{w_i\}$ for the space of holomorphic 1- forms such that $\int_{A_i} w_j = \delta_{ij}$, we can define the period matrix $\Omega = \left[\int_{B_i} w_j\right]$ of $\aleph$. It can be shown that Ω is an element of the Siegel upper-half space $\mathcal{H}_g$. For any $\tau \in \mathcal{H}_g$ and any $z \in \mathbb{C}^g$ the Riemann's theta function is defined as $\theta(z,\tau) = \sum_{u \in \mathbb{Z}^g} e^{\pi i (u^t \tau u + 2u^t z)}$. Any point $e \in Jac(\aleph)$, where $Jac(\aleph)$ is the Jacobian of the curve $\aleph$ can be written uniquely as $e = (b,a)\begin{bmatrix} 1_g \\ \Omega \end{bmatrix}$, where $a, b \in \mathbb{R}^g$. For any $a, b \in \mathbb{Q}^g$, the theta function with rational characteristics is defined as $\theta\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau) = \sum_{u \in \mathbb{Z}^g} e^{\pi i((u+a)^t \tau (u+a) + 2(u+a)^t (z+b))}$. When the entries of column vectors a and b are from the set $\left\{0 \frac{1}{2}\right\}$, then the corresponding theta functions with rational characteristics are known as theta characteristics. A scalar obtained by evaluating a theta characteristics at $z = 0$ is called a thetanull.

The problem of expressing branch points $\pi : \aleph \to \mathbb{P}^1$ in terms of transcendentals (period matrix, thetanulls, etc.,) is classical. This is an old problem that goes back to Riemann, Jacobi, Picard and Rosenhein. We do not aim here at a complete account of the classical or contemporary work on these problems. We determine the curves of genus 4 in terms of thetanulls and further study relations among the classical thetanulls of cyclic curves $\aleph$ (of genus 4) with an automorphisms. In our work we use formulas for small genus curves introduced by Rosenhein, Thomae's formulas for hyperelliptic curves, some recent results of Hurwitz space theory, and symbolic manipulation. Inverting the period map has an application in fast genus two curves arithmetic in cryptography. We determine similar formulas for genus 4 hyperelliptic curves as the one used in cryptography using genus 2 algebraic curves.

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