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PAPER

Roots of a cubic and simple proof of Fermat's last theorem for n=3

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Introduction:
Fermat's last theorem (FLT) which was written in 1637, became public in 1670, without proof. It has not only evoked the interest of mathematicians but baffled many for over three hundred and fifty years [1],[2]. It is well known that FLT despite its rather simple statement had been difficult to prove even for the small prime exponent $n=3$ .The main objective of this paper is to provide simple proof of the theorem for this special exponent $n=3$ using the Method of Taragalia and Cardom of solving a cubic which is much older than Fermat's last theorem.

Proof of Fermat's last theorem for n=3
Fermat's last theorem for $n=3$ can be stated as that the equation
$z^3 = y^3 + x^3, \quad (x,y)=1$
(1.1)
is not satisfied by non-trivial integer triples x,y,z.
Assume that the equation is satisfied by non-trivial integer triples x,y,z.
If $xyz \not\equiv 0 (\bmod 3)$, let

$$x = 3m \pm 1, \quad y = 3k \pm 1, \quad z = 3s \pm 1,$$

then we have

$$y^3 + x^3 \equiv (2,0,-2)(\bmod 3^2)$$

However $z^3 \equiv \pm 1 (\bmod 3^2)$, therefore our assumption is wrong and we conclude that $xyz \equiv 0 (\bmod 3)$.
Since we consider the equation (1.1) for positive and negative integer values, without loss of generality, we can assume that $y \equiv 0(\bmod 3)$, and let $y \equiv 0 (\bmod 3^m)$.
Then $z - x = 3^{3m-1}u^3, \quad z - y = h^3, \quad x + y = g^3$ due to Barlow relations, where $h, 3^m u$ and g are respectively the factors of x, y and z. Now,

$$g^3 - 3^{3m-1}u^3 - h^3 = 2(x + y - z) \tag{1.2}$$

Since

$$x + y - z = x - (z - y) = (x + y) - z = y - (z - x),$$

$$\therefore \quad x + y - z \equiv 0(3^m ugh)$$

It can be shown that [1],

$$(x+y-z)^3 = 3(x+y)(z-x)(z-y) = 3^{3m}h^3u^3g^3 \tag{1.3}$$

and therefore

$$g^3 - h^3 - 3^{3m-1}u^3 - 2\cdot 3^m ugh = 0$$

(1.4)

This necessary condition must be satisfied by the integer parameters $h, 3^m u$, $and\ g$ of $x, y\ and\ z$ respectively. We will first fix the parameters $3^m u$ of y and show that (1.4) is not satisfied by integer g for any integer h using the Method of Tartagalia and Cardoon[4] of finding a roots of a cubic. The equation (1.4) is of the form

$$g^3 - 3vwg - v^3 - w^3 = 0 \tag{1.5}$$

where $v^3 + w^3 = 3^{3m-1}u^3 + h^3$, $vw = 2.3^{m-1}uh$, and its roots can be written as

$$v+w, v\omega + w\omega^2 \text{ and } v\omega^2 + w\omega \tag{1.6}$$

with the cube root ω of unity. Now v^3, w^3 are the roots of the equation

$$t^2 + Gt - H^3 = 0 \tag{1.7}$$

where $G = -(3^{3m-1}u^3 + h^3)$, $H = -2.3^{m-1}uh.$ Moreover v, w real and distinct and this equation has only one real root, namely, $g = v + w$, since the discriminant of (1.5) is -27Δ,where $\Delta = (G^2 + 4H^3)$ is the discriminant of (1.7),and it is negative. Therefore, it has only one real root [4]. Note that

$\Delta = (G^2 + 4H^3) = 3^{6m-2}u^6 - 14.3^{3m-3}u^3h^3 + h^6 = (3^{3m-1}u^3 - h^3)^2 + 4.3^{3m-3}u^3h^3$

which is positive when $uh > 0$. On the other hand $.\Delta = (3^{3m-1}u^3 + h^3) - 32.3^{3m-3}u^3h^3$ which is positive when $uh < 0$.

If

$$v_1 + w_1, v_1\omega + w_1\omega^2, v_1\omega^2 + w_1\omega \tag{1.8}$$

is another representation of the roots, we must have

$v_1 + w_1 = v + w, v_1\omega + w_1\omega^2 = v\omega + w\omega^2, v_1\omega^2 + w_1\omega = v\omega^2 + w\omega$

or $\quad v_1 + w_1 = v + w, v_1\omega + w_1\omega^2 = v\omega^2 + w\omega, v_1\omega^2 + w_1\omega = v\omega + w\omega^2$

In other words $v_1 - v = w - w_1, (v_1 - v) = (w - w_1)\omega$ or $(v_1 - w) = (v - w_1), (v_1 - w) = (v - w_1)\omega$

This means that

$v_1 = v, w_1 = w$ or $v_1 = w, w_1 = v.$

Therefore, roots must be unique. In particular, we must have a unique real root. From (1.4), it follows that the real root can be expressed in the form

$g - h = 3^{m-1}j$ or $g = 3^{m-1}j + h$

where j is an integer satisfying $(3, j) = 1$. This is due to the fact that

$$g^3 - h^3 = (g-h)[(g-h)^2 + 3gh] \equiv 0 (\bmod 3^m).$$

It is now clear that h^3 satisfies the equation $t^2 + Gt - H^3 = 0$, This means that $h^6 - (3^{3m-1}u^3 + h^3)h^3 + 8.3^{3m-3}u^3h^3 = 0$ implying that $u = 0$ or $h = 0$, that is $y = 0$ or $x = 0$ in Fermat's equation. Hence there is no non-trivial integral triple satisfying the Fermat equation (1.1).

We have shown that Fermat's Last Theorem for $n = 3$ can be proved without depending on the method of infinite descent or complex analysis. The proof given above is short and simple, and simpler than proving [3] the theorem using the method of infinite descent.

References

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