4.16 Structure of primitive Pythagorean triples and the proof of a Fermat's theorem

W.J.M.L.P. Jayasinghe, R.A.D. Piyadasa
Department of mathematics, University of Kelaniya

ABSTRACT

In a short survey of survey of primitive Pythagorean triples (x, y, z) $0 < x < y < z$, we have found that one of x, y, z is divisible by 5 and z is not divisible by 3, there are Pythagorean triples whose corresponding element are equal , but there cannot be two Pythagorean triples such that $(x_1, y_1, z_1), (x_1, z_1, z_2)$, where z_1 and z_2 hypotenuses of the corresponding Pythagorean triples. This is due to a Fermat's theorem [1] that the area of a Pythagorean triangle cannot be a perfect square of an integer, which can directly be used to prove Fermat's last theorem for $n = 4$. Therefore the preceding theorem is proved using elementary mathematics, which is the one of the main objectives of this contribution. All results in this contribution are summarized as a theorem.

Theorem

If (x, y, z) is a primitive Pythagorean triangle, where z is the hypotenuse, then z is never divisible by 3, and $xy \equiv 0 (\mathrm{mod}\, 3)$, $xyz = 0 (\mathrm{mod}\, 5)$,and there are Pythagorean triangles whose corresponding one side is the same. But there are no two Pythagorean triangles such that (x_1, y_1, z_1) , (x_1, z_1, z_2)), where z_1, z_2 are hypotenuses.

Proof of the theorem

Pythagoras' equation can be written as

$$z^2 = y^2 + x^2, (x, y) = 1 \tag{1}$$

and if $z \equiv 0 (\mathrm{mod}\, 3)$,then since xy is not divisible by 3, $z^2 = y^2 - 1 + x^2 - 1 + 2$ .Now, it follows at once from Fermat's little theorem that z cannot be divisible by 3. If xyz is not divisible by 5, squaring (1), one obtains $z^4 = y^4 + x^4 + 2x^2 y^2$ and hence $z^4 - 1 = y^4 - 1 + x^4 - 1 + 2(x^2 y^2 \pm 1) + t$, where $t = -1$ or 3. Therefore $xyz \equiv 0 (\mathrm{mod}\, 5)$. It is easy to obtain two Pythagorean triples whose corresponding two elements are equal, from the pair-wise disjoint sets which have recently been obtained in Ref.2. For example $365^2 = 364^2 + 27^2, 365^2 = 364^2 + 27^2, 365^2 = 357^2 + 76^2$ .Now, assume that there exists two primitive Pythagorean triples of the form

$$a^2 = b^2 + c^2 \tag{1}$$

$$d^2 = a^2 + c^2 \tag{2}$$

It is clear that a is odd and $c \equiv 0 (\mathrm{mod}\, 3)$. From these two equations, one obtains immediately $d^2 - b^2 = 2c^2$, $d^2 + b^2 = 2a^2$, and therefore

$$d^4 - b^4 = 4c^2 a^2 = w_0^2 \tag{3}$$

It has been proved by Fermat, after obtaining the representation of the primitive Pythagorean triples as $x = 2rs, y = r^2 - s^2, z = r^2 + s^2$, where $0 < s < r$ and r, s are of opposite parity, that (3) has no non trivial integral solution for d, b, w. To prove the same in an easy manner consider the equation $d^2 + b^2 = 2a^2$ in the form $d^2 - a^2 = a^2 - b^2$ and writing it as $(d - a)(d + a) = (a - b)(a + b)$ use the technique used in Ref.3 to obtain the parametric solution for d and b If $d - a = a - b$., then $d + b = 2a$, from we deduce $db = a^2$. This never holds since $(d, a) = 1 = (b, a)$ by (1) and (2). If $(d - a)\frac{u}{v} = (a - b)$, where $(u, v) = 1$, then $(d + a)\frac{v}{u} = (a + b)$. From these two relations, one derives the simultaneous equations

$$vd - ub = a(u - v) \qquad (4a)$$

$$ud + vb = a(u + v) \qquad (4b)$$

From (4a),(4b),it is easy to deduce the relations that we need to prove the theorem as $(v^2 + u^2)d = [2uv + u^2 - v^2]a$, $(v^2 + u^2)b = [2uv - (u^2 - v^2)]a$, $(v^2 + u^2)(d + b)) = 4uva$, $v^2 + u^2 = 2a$, assuming that u and v are odd.

Hence $d - b = (u^2 - v^2), (d + b) = 2uv$. Therefore $d^2 - b^2 = 2(u^2 - v^2)uv = 2c^2$ and hence u, v are perfect squares and we can find two integers g, h such that. $g^4 - h^4 = w_1^2 < w_0^2$. Now, proof of the last part of the theorem follows from the method of infinite descends of Fermat. Even if u and v are of opposite parity proof of the theorem can be done in the same way.

To complete the proof of a Fermat's theorem that $g^4 - h^4 = w_0^2$ is not satisfied by any non-trivial integers, we write $(g^2 + h^2)(g^2 - h^2) = w_0^2$, where g, h are of opposite parity, to obtain $g^2 + h^2 = x^2, g^2 - h^2 = y^2$ and $x^4 - y^4 = 4g^2h^2 = z_0^2$, where x and y are odd and co-prime. But, in the case of the main theorem, we have shown that this is not satisfied by any non-trivial odd x, y and even z_0 numbers .This completes the proof of the Fermat's theorem we mentioned above.

References

(1)Paulo Rebenboim, Fermat's last theorem for amateurs, Springer, Verlag (1991)

(2)Piyadasa R.A.D. et.al ,10th international conference of Sri Lanka studies, (2005), Abstract ,pp164

(3) Piyadasa R.A.D. Analytical solution of Fermat's last theorem for $n = 4$,