4.17 Singularities of the elastic S-matrix element

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ABSTRACT

It is well known that the standard conventional method of integral equations is not able to explain the analyticity of the elastic S-matrix element for the nuclear optical potential including the Coulomb potential. It has been shown[1],[2] that the cutting down of the potential at a large distance is essential to get rid of the redundant poles of the S-matrix element in case of an attractive exponentially decaying potential. This method has been found [3] to be quite general and it does not change the physics of the problem. Using this method , analiticity and the singularities of the S-matrix element is discussed.

Singularities of the elastic S-matrix element

Partial wave radial wave equation of angular momentum l corresponding to elastic scattering is given by,

$$\left[\frac{d^2}{dr^2}+k^2-\frac{l(l+1)}{r^2}\right]u_l(k,r)=\frac{2\mu}{\hbar^2}\left[V(r)+V_c(r)+iW(r)\right]u_l(k,r) \tag{1}$$

where $V(r)$ is the real part of nuclear potential, $W(r)$ is the imaginary part of the optical potential $V_c(r)$ is the Coulomb potential, and k is the incident wave number. Energy dependence of the optical potential is usually through laboratory energy E_{lab} and hence it depend on k^2 and therefore $k^2-\frac{2\mu}{\hbar^2}\left[V(r)+V_c(r)+i\,W(r)\right]$ is depending on k through k^2. It is well known that $V_c(r)$ is independent of k .In order to make $u_l(k,r)$ an entire function of k, we impose k independent boundary condition at the origin . Now, we can make use of a well known theorem of Poincare to deduce that the wave function is an entire function of k^2 and hence it is an entire function of k as well. We cut off the exponential tails of the optical potential at sufficiently large R_m and use the relation

$$\frac{1}{u_l}\frac{du_l}{dr}=\frac{u_l'^{(-)}(k,r)-S_l^n(k,R_m)\,u_l'^{(+)}(k,r)}{u_l^{(-)}(k,r)-S_l^n(k,R_m)\,u_l^{(+)}(k,r)} \qquad \text{for } r\geq R_m \tag{2}$$

to define $S_l^n(k,R_m)$,where $u_l^{(-)}(k,r)$ and $u_l^{(+)}(k,r)$ stand for incoming and outgoing Coulomb wave functions respectively which are given by

$$u_l^{(\pm)}(k,r)=\pm i\left[\frac{\Gamma(l+1+i\eta)}{\Gamma(l+1-i\eta)}\right]^{\frac{1}{2}} e^{\left[\frac{\pi\eta}{2}\mp i(l+1)\frac{\pi}{2}\right]}\,W_{\mp i\eta,\,l+\frac{1}{2}}(-2ikr) \tag{3}$$

where W are the Whittaker functions. In the limit $R_m\to\infty$ $S_l^n(k,R_m)$,the nuclear part of the S-matrix element , becomes $S_l^n(k)$ and the redundant poles removed[1],[2].Now, the nuclear S-matrix element , in terms of the Whittaker functions is given by

$$S_l^n(k) = (-1)^l \frac{\Gamma(l+1-i\eta)}{\Gamma(l+1+i\eta)} \frac{W'_{i\eta,l+\frac{1}{2}}(2ikr) - P_l(k,r)\, W_{i\eta,l+\frac{1}{2}}(2ikr)}{W'_{-i\eta,l+\frac{1}{2}}(-2ikr) - P_l(-k,r) W_{-i\eta,l+\frac{1}{2}}(-2ikr)} \quad ,r \geq R_m \qquad (4)$$

where $P_l(k,r) = \frac{u_l'(k,r)}{u_l(k,r)}$,and $S_l^n(k)$ has an essential singularity at $k = 0$, which is apparent from the Wister's definition of the Gamma function $\Gamma(z)$ since $z = l+1 \pm i\eta$ .However, this singularity has no any physical meaning and is an outcome of treating $\frac{2\eta k}{r}$ as well defined quantity for all k including $k = 0$ in the corresponding Schrödinger equation .The infinite number of zeros and poles of S – matrix element due to the Gamma functions associated with S – matrix element have to be interpreted carefully. $S_l^\eta(k) = 0$ at the zeros of $\frac{1}{\Gamma(l+1+i\eta)}$ and then the total wave function reduces to

$$u_l^{(-)}(k,r) = -i \left[\frac{\Gamma(l+1-i\eta)}{\Gamma(l+1+i\eta)}\right]^{\frac{1}{2}} e^{\left[\frac{\pi\eta}{2} + i(l+1)\frac{\pi}{2}\right]} W_{i\eta,l+\frac{1}{2}}(2ikr)$$

which is also zero. Even though the corresponding energies of these states are negative since the corresponding wave number is given by

$$k = i\frac{z_1 z_2 e^2}{\hbar^2 (n+l+1)} \qquad n = 0, 1, 2, ...$$

they are not physically meaningful bound states as found in[1],[2] long ago. These states are unphysical since poles are redundant poles. This fact is clearly understood by the fact that all these poles are absent in the physically meaningful total S – matrix element.

For large $|k|$, $S_l^\eta(k) \sim (-)^l e^{-2ikr} S(k)$, where $S(k) = \frac{[-ik + P_l(k)]}{[ik - P_l(-k)]}$. since $W = e^{\pm 2ikr}$ for large k . Therefore the S-matrix element has an essential singularity at infinity, which is on the imaginary axis. It is clear that there are no redundant poles in the total S-matrix element is free from redundant poles since $S_l^t(k) = S^C S_l^n$, where $S^C = \frac{\Gamma(l+1+i\eta)}{\Gamma(l+1-i\eta)}$.

References

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(2) Ma S. T., Phys. Rev. 71, 195 (1946)
(3) Barut A.O. et.al Jour. of Math. Phys. 2, 178 (1961)