# 4.24 A general relativistic solution for the space time generated by a spherical shell with constant uniform density

N.A.S.N. Wimaladharma*, Nalin de Silva
Department of Mathematics,University of Kelaniya

## ABSTRACT

In this paper we present a general relativistic solution for the space time generated by a spherical shell of uniform density. The Einstein's field equations are solved for a distribution of matter in the form of a spherical shell with inner radius $a$ and outer radius $b$ and with uniform constant density $\rho$.

We first consider the region which contains matter ($a < r < b$). As the metric has to be spherically symmetric we take the metric in the form $ds^2 = e^{\nu} c^2 dt^2 - e^{\lambda} dr^2 - r^2 d\Omega^2$, where $d\Omega^2 = \left(d\theta^2 + \sin^2\theta d\phi^2\right)$, $\lambda$ and $\nu$ are functions of $r$ as in Adler, Bazin and Schiffer[1] where the space time metric for a spherically symmetric distribution of matter in the form of sphere of uniform density has been worked out.

Solving the field equations, we obtain

$$e^{\lambda} = \frac{1}{\left(1 - \frac{r^2}{R^2} + \frac{E}{r}\right)} \quad \text{and} \quad e^{\nu} = \left(1 - \frac{r^2}{R^2} + \frac{E}{r}\right)\left(A + \frac{B}{2}\int \frac{r}{\left(1 - \frac{r^2}{R^2} + \frac{E}{r}\right)^{3/2}} dr\right)^2$$

Here $R^2 = \frac{3c^2}{8\pi\kappa\rho}$, where $c$ and $\kappa$ are the velocity of the light and the gravitational constant respectively and $A$, $B$ and $E$ are constants to be determined.

Let the metric for the matter free regions be $ds^2 = e^{\nu} c^2 dt^2 - e^{\lambda} dr^2 - r^2 d\Omega^2$, where as before from spherical symmetry $\lambda$ and $\nu$ are functions of $r$.

Solving the field equations, we obtain, $e^{\lambda}$ and $e^{\upsilon}$ in the form

$e^{\lambda} = \frac{1}{\left(1 + \frac{G}{r}\right)}$ and $e^{\upsilon} = D\left(1 + \frac{G}{r}\right)$, for the regions $0 < r < a$, and $r > b$. where $D$ and $G$ are constants.

For the region $0 < r < a$, when $r = 0$, the metric should be regular. So $G = 0$. Hence the metric for the region $0 < r < a$ is $ds^2 = Dc^2dt^2 - dr^2 - r^2 d\Omega^2$.

For the region $r > b$, the metric should be Lorentzian at infinity. So $D = 1$. Hence the metric for the exterior matter free region is

$$ds^2 = \left(1 + \frac{G}{r}\right)c^2 dt^2 - \frac{1}{\left(1 + \frac{G}{r}\right)} dr^2 - r^2 d\Omega^2 .$$

Then we can write the metric for the space-time as

$$ds^2 = D c^2 dt^2 - dr^2 - r^2 d\Omega^2 \qquad \text{,when } 0 < r < a ,$$

$$ds^2 = \left(1 - \frac{r^2}{R^2} + \frac{E}{r}\right)\left(A + \frac{B}{2}\int \frac{r}{\left(1 - \frac{r^2}{R^2} + \frac{E}{r}\right)^{3/2}} dr\right)^2 c^2 dt'^2 - \frac{1}{\left(1 - \frac{r^2}{R^2} + \frac{E}{r}\right)} dr^2 - r^2 d\Omega^2 ,$$

when $a \le r \le b$ , and

$$ds^2 = \left(1 + \frac{G}{r}\right)c^2 dt^2 - \frac{1}{\left(1 + \frac{G}{r}\right)} dr^2 - r^2 d\Omega^2 \quad , \qquad \text{when } r > b.$$

Applying the boundary conditions at $r = a$ and $r = b$, we have

$$E = \frac{a^3}{R^2} \quad , \qquad G = \frac{-\left(b^3 - a^3\right)}{R^2} \quad ,$$

$$A = \frac{\left(b^3 - a^3\right)\left(\frac{R^2}{3} - \frac{1}{2}\int \frac{r}{\left(1 - \frac{r^2}{R^2} + \frac{a^3}{R^2 r}\right)^{3/2}} dr \Bigg|_{r=a}\right)}{\left(\frac{b^3 R^2}{\left(1 - \frac{b^2}{R^2} + \frac{a^3}{R^2 b}\right)^{1/2}} - \left(2b^3 + a^3\right)\left(\frac{R^2}{3} + \frac{1}{2}\int_a^b \frac{r}{\left(1 - \frac{r^2}{R^2} + \frac{a^3}{R^2 r}\right)^{3/2}} dr\right)\right)} \qquad \text{(i)}$$

$$B = \frac{\left(b^3 - a^3\right)}{\left(\frac{b^3 R^2}{\left(1 - \frac{b^2}{R^2} + \frac{a^3}{R^2 b}\right)^{1/2}} - \left(2b^3 + a^3\right)\left(\frac{R^2}{3} + \frac{1}{2}\int_a^b \frac{r}{\left(1 - \frac{r^2}{R^2} + \frac{a^3}{R^2 r}\right)^{3/2}} dr\right)\right)} \qquad \text{(ii)}$$

where

$$\int \frac{r}{\left(1-\frac{r^2}{R^2}+\frac{a^3}{R^2 r}\right)^{3/2}}dr = \frac{2\left(a^3-r^3+rR^2\right)^2\left(-9a^6\sqrt{r}-3a^3r^{3/2}R^2+2r^{5/2}R^4\right)}{r^{2/3}\left(\frac{a^3-r^3+rR^2}{rR^2}\right)^{3/2}\left(-27a^9+27a^6r^3-27a^6rR^2+4a^3R^6-4r^3R^6+4rR^6\right)}$$

$$+\frac{1}{r^2\left(\frac{a^3-r^3+rR^2}{rR^2}\right)^{3/2}\left(27a^6-4R^6\right)}\left(2\left(a^3-r^3+rR^2\right)^{3/2}\sqrt{a^3+r^3\left(-1+\frac{R^2}{r^2}\right)}\right)$$

$$+\frac{1}{r\left(\frac{a^3-r^3+rR^2}{rR^2}\right)}\left(a^3\sqrt{\frac{\left(-\frac{1}{r}+r_2\right)\left(\frac{1}{r}-r_3\right)\left(r_1-r_3\right)}{\left(r_2-r_3\right)^2\left(r_1-r_3\right)}}\left(F(\phi/m)\right)+\left(9a^3+2R^4r_1\right)\right.$$

$$\left.+2R^4\,E(\phi/m)\left(-r_1+r_3\right)\right)))).$$

$$F(\phi/m)=\int_0^{\phi}\left(1-m\sin^2\theta\right)^{-1/2}d\theta \quad \text{and} \quad E(\phi/m)=\int_0^{\phi}\left(1-m\sin^2\theta\right)^{1/2}d\theta\ ,\ -\frac{\pi}{2}<\phi<\frac{\pi}{2}$$

are the Elliptic integrals of first kind and second kind respectively, where

$$\phi=\mathrm{Arcsin}\left[\sqrt{\frac{\left(-\frac{1}{r}+r_3\right)}{\left(r_3-r_2\right)}}\right] \quad \text{and} \quad m=\frac{\left(r_2-r_3\right)}{\left(r_1-r_3\right)}.$$

Here $r_1$ = The first root of $\left(-1+R^2r^2+a^3r^3\right)$.

$r_2$ = The second root of $\left(-1+R^2r^2+a^3r^3\right)$.

$r_3$ = The third root of $\left(-1+R^2r^2+a^3r^3\right)$.

Furthermore we know that the potential $\phi$ of a shell of inner radius $a_1$ and outer radius $b_1$ and constant uniform density in Newtonian gravitation is given by

$$\phi=2\pi\kappa\rho\left(a_1^{\,2}-b_1^{\,2}\right) \qquad 0<r<a_1$$

$$\phi=\frac{2\pi\kappa\rho}{3}r^2+\frac{4\pi\kappa\rho}{3r}a_1^{\,3}-2\pi\kappa\rho b_1^{\,2} \qquad a_1\le r\le b_1$$

$$\phi=-\frac{4\pi\kappa\rho}{3r}\left(a_1^{\,3}-b_1^{\,3}\right) \qquad b_1<r\ .$$

Using the fact that $g_{00} \cong \left(1+\frac{2\phi}{c^2}\right)$, (for example in Adler, Bazin and Schiffer[1] )we find that the constants $a_1, b_1$ in Newtonian gravitation and $D$ can be written in the form

$$a_1 = a\left(\frac{(b^3-a^3)/3}{\frac{b^3}{\left(1-\frac{b^2}{R^2}+\frac{a^3}{R^2 b}\right)^{1/2}}-\frac{(2b^3+a^3)}{3}}\right)^{2/3} \qquad \text{(iii)}$$

$$b_1 = R\sqrt{\frac{2}{3}}\left(1-\left(\frac{(b^3-a^3)/3}{\frac{b^3}{\left(1-\frac{b^2}{R^2}+\frac{a^3}{R^2 b}\right)^{1/2}}-\frac{(2b^3+a^3)}{3}}\right)^2\right)^{1/2} \qquad \text{(iv)}$$

$$D = \left(1+\frac{3(a_1^2-b_1^2)}{2R^2}\right).$$

Hence the final form of the metric is

$$ds^2 = \left(1+\frac{3(a_1^2-b_1^2)}{2R^2}\right)c^2dt^2 - dr^2 - r^2 d\Omega^2 \qquad 0<r<a$$

$$ds^2 = \left(1-\frac{r^2}{R^2}+\frac{a^3}{R^2 r}\right)\left(A+\frac{B}{2}\int\frac{r}{\left(1-\frac{r^2}{R^2}+\frac{a^3}{R^2 r}\right)^{3/2}}dr\right)^2 c^2dt'^2 - \frac{1}{\left(1-\frac{r^2}{R^2}+\frac{a^3}{R^2 r}\right)}dr^2 - r^2 d\Omega^2$$

$$a \le r \le b$$

$$ds^2 = \left(1-\frac{(b^3-a^3)}{R^2 r}\right)c^2dt^2 - \frac{1}{\left(1-\frac{(b^3-a^3)}{R^2 r}\right)}dr^2 - r^2 d\Omega^2 \qquad b<r.$$

where $A, B, a_1$ and $b_1$ are given by the equations (i), (ii), (iii) and (iv) respectively.

References

1. Adler R., Bazin M., and Schiffer M., Introduction to General Relativity, McGraw-Hill Inc, New York (1965).