4.5 Analytical Proof of Fermat's Last Theorem for $n=3$.

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Dedicated to late Prof. S.B.P. Wickeramasuriya

ABSTRACT

1. Introduction

It is well known that the proof of Fermat's Last Theorem, in general, is extremely difficult. It is surprising that the proof of theorem for $n=3$, the smallest corresponding number, given by Leonard Euler, which is supposed to be the simplest, is also difficult and erroneous. Paulo Rebenboin claims that he has patched up [1] the Euler's proof, which is very difficult to understand, however. In this article we present a simple and short proof of the Fermat's last theorem. .

Fermat's Last Theorem

The equation

$$z^n = y^n + x^n, \quad (x, y) = 1$$

has no nontrivial integral solutions (*x,y,z*) for any prime $n \geq 3$.

2. Proof of the Fermat's last theorem for $n = 3$

In the following, the parametric solution to the problem based on very simple three lemmas is given.

2.1 Lemma

If $a^3 - b^3$ is divisible by $3^{\mu}(\mu \neq 0)$ and $(a, 3) = 1 = (b, 3)$, then $(a-b)$ is divisible by $3^{\mu-1}$ and $\mu \geq 2$.

This lemma can be easily proved substituting $a-b=k$ in $a^3 - b^3$ and we assume it without proof.

2.2 Lemma

If the equation

$$z^3 = y^3 + x^3, \quad (x, y) = 1 \tag{1}$$

has a non trivial integral solution (x, y, z), then one of x, y, z is divisible by 3. Proof of this lemma is also simple and we assumed it without proof.

Now, (1) takes the form

$$z^3 = 3^{3\beta}\gamma^3 + x^3, \quad (3, x) = 1 \tag{2}$$

2.3 Lemma

There are two integers α and β such that

$$z - x = 3^{3\beta-1}\alpha^3, \quad (3, \alpha) = 1.$$

Proof of this lemma is exactly the same as in the case of analytic solution of Pythagoras' theorem [2] and let us assume it without proof.

Now (2) takes the form

$$z^3 = 3^{3\beta}\alpha^3\eta^3 + x^3, \quad (3, z) = 1 \tag{3}$$

From the equation $3ys(y+s) = x^3 - s^3$, it follows that s should be of the form δ^3, where $(\delta, 3) = 1$ since s divides x and $(s, y) = 1$. Also note also that $\delta^3 = z - y$. Then the above equation becomes

$$3y\delta^3\left(y + \delta^3\right) = x^3 - \delta^9 \tag{4}$$

Now it is clear that $x-\delta$ is divisible by three. Let us consider the expression $x+y-z$. $x+y-z=x-(z-y)=x-\delta^3$. Now consider the original equation $z^3=y^3+x^3$, $(x,y)=1$ in the form that $z^3=(x+y)\left((x+y)^2-3xy\right)$. It is clear that $x+y$ and the term, $(x+y)^2-3xy$ are co-prime and therefore $x+y=\theta^3$, where $z=\theta\xi$ and $(\theta,\xi)=1$. Now again $x+y-z=\theta^3-\theta\xi=\theta(\theta^2-\xi)$ and therefore $x-\delta^3$ is divisible θ. $x+y-z=y-(z-x)=3^\beta\alpha\eta-3^{3\beta}\alpha^3=3^\beta\alpha\left(\eta-3^{2\beta-1}\alpha^2\right)$ and therefore $x-\delta^3$ is divisible by $3^\beta\alpha$. x is divisible by δ, which follows from (4) and therefore $x-\delta^3$ is divisible by $3^\beta\alpha\theta\delta$. Now consider (4) in the form $3^{\beta+1}\alpha\eta\delta^3\theta\xi=\left(x-\delta^3\right)\left(x^2+x\delta^3+\delta^9\right)$. From which one understands that $x-\delta^3=3^\beta\alpha\theta\delta$ and since $z-x=3^{3\beta-1}\alpha^3$.

$$x=3^\beta\alpha\theta\delta+\delta^3 \quad \text{(a)}$$

$$y=3^\beta\alpha\theta\delta+3^{3\beta-1}\alpha^3 \quad \text{(b)}$$

$$z=3^{3\beta-1}\alpha^3+3^\beta\alpha\theta\delta+\delta^3 \quad \text{(c)}$$

In addition to this, we have $x+y=\theta^3$ and $\left(\eta-3^{2\beta-1}\alpha^2\right)=\theta\delta$, and therefore substituting for η in y, we get

$$\theta^3-\delta^3-2.3^\beta\alpha\theta\delta-3^{3\beta-1}\alpha^3=0 \quad \text{(d)}$$

Therefore by lemma (1), $\theta-\delta$ should be divisible $3^{\beta-1}$. Expressing $3^{3\beta-1}\alpha^3$ as $8.3^{3\beta-3}\alpha^3+3^{3\beta-3}\alpha^3$ and $\theta=3^{\beta-1}g+\delta$,we obtain from (d) that

$$(g-2\alpha)\left(\delta^2+3^{\beta-1}g\delta+3^{2\beta-3}\left(g^2+2\alpha g+4\alpha^2\right)\right)=3^{2\beta-3}\alpha^3 \quad (5)$$

If $X=(g-2\alpha)$, $Y=\left(\delta^2+3^{\beta-1}g\delta+3^{2\beta-3}\left(g^2+2\alpha g\right)+4\alpha^2\right)$, then it is clear that $(3,Y)=1$,

Now we prove the Fermat's last theorem for $n=3$, showing that (d) is never satisfied. If $\alpha=1$, $g=2+3^{2\beta-3}$ and $Y=1=(\delta+\frac{3^{\beta-1}g}{2})^2+3^{2\beta-3}(\frac{g^2}{4}+2g+4)$ which is never satisfied.

Similarly , the proof of the theorem follows when $\alpha=-1$ since $\beta\geq 2$. If $\alpha\neq 1$, $g=2\alpha.+3^{2\beta-3}q^3$ and $\theta=2\alpha 3^{\beta-1}+(3^{3\beta-4}p^3+\delta)$. The equation (d) is of the form

$$\theta^3-3(2.3^{\beta-1}\alpha)\delta\theta-8.3^{3\beta-3}\alpha^3-(3^{3\beta-3}\alpha^3+\delta^3)=0 \quad (6)$$

and is also of the form.

$x^3-3.u.vx-u^3-v^3=0$ and therefore we can make use of the well known method of Tatagliya and Cardon (see[3]).Then u^3 must be a solution of the quadratic $x^2+Gx-H^3=0$ and the roots of (d) for θ are $u+v, u\varpi+v\varpi^2, u\varpi^2+v\varpi$, where ϖ is the cube root of unity. It can be easily shown that this occurs only if $u=0$. which gives $\alpha=0$ and this corresponds to the trivial solution $x.y.z=0$.Hence the proof of the theorem.

References:

1) Paulo Ribenboim, Fermat;s last theorem for amateurs, Springer 1991.
2) Piyadasa R.A.D. & Karunatileke N.G.A., 10th International Conference of Sri Lanka Studies Abstract pp 108, 2005.
3) Archfold J.W. Algebra Sir Issac Pitman & Sons ,LTD London, ,pp.174, 1961.