4.6 Mean Value Theorem and Fermat's Last Theorem for n=3

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Dedicated to late Prof. S.B.P. Wickramasuriya

ABSTRACT

Fermat's last theorem can be stated as that the equation

$$z^n = y^n + x^n, (x, y) = 1 \quad \text{(A)}$$

has no non-trivial integral solutions for (x, y, z) except for n=2, and therefore we have carefully examined all primitive Pythagorean triples [1],[2] and we solved Pythagoras' equation analytically resulting in a new generators for Primitive Pythagorean triples and a simple conjecture which is explained and proved for $n = 3$ in the following.

1. Conjecture

When y is divisible by 2, all primitive Pythagorean triples (x,y,z) are related by

$$z^2 - x^2 = y^2 = 2.(z-x)(x + \theta h) \quad (1)$$

where $h = z - x = 2^{2\beta-1}\alpha^2$ and $\theta = \frac{1}{2}$ .This resembles the Mean value theorem

$f(z) - f(x) = (z - x).f'(\xi)$, where $f(x) = x^2$, $f'(\xi) = 2.\xi$ and $\xi = \frac{z + x}{2}$,which is a perfect square. If (A) has a non trivial integral solutions for a prime $n \neq 2$, then there may be such solutions that one of x, y, z is divisible by n which is well known .Then the equation (A) can be put into the form

$$z^n - x^n = n.n^{\beta n-1}\alpha^n.\xi^{n-1} = n^{\beta n}\alpha^n\gamma^n \quad (2)$$

with $z - x = n^{\beta n-1}\alpha^n$ assuming that y is divisible by n ,where all letters except ξ stand for integers.. This also resembles the Mean vale theorem

$f(z) - f(x) = (z - x).f'(\xi)$ with $f(x) = x^n$. It is conjectured [1]that if (A) is true, then $\gamma^n = \xi^{n-1}$ does not hold with integral γ, ξ except for $n = 2$.

2.Proof of the Conjecture

In this contribution, the conjecture in the previous section is proved for $n = 3$. Proof is based on the following lemmas.

2.1 Lemma

If

$$z^3 = y^3 + x^3, (x, y) = 1 \quad (3)$$

has an integral solution for x, y, z ,then one of x, y, z is divisible by 3.

The proof of this lemma is simple and it is assumed without proof.

Since the above equation holds for $-x, -y, -z$, without loss of generality, one may assume that y is divisible by 3.

2.2 Lemma

If y is divisible by 3, then $z - x = 3^{3\beta-1}\alpha^3$, where α is an integer including ± 1. The proof of this lemma is exactly the same as as in the case of analytic solution of the Pythagoras' equation for primitive Pythagorean triples and is also assumed without proof.

Now, (3) takes the form

$$z^3 = 3^{3\beta}\alpha^3\gamma^3 + x^3, (x, y) = 1 \tag{4}$$

2.3 Lemma

If $(a,3) = 1 = (b,3)$, then it follows from the Fermat's little theorem that $a^3 \pm b^3$ is divisible by 3 and since $a^3 \pm b^3 = (a \pm b)((a \pm b)^2 \pm 3ab)$, the least power of 3 that divides $a^3 \pm b^3$ is 2.

Substituting $z - x = 3^{3\beta-1}\alpha^3$ in (4), one obtains

$$\gamma^3 = 3^{6\beta-3}\alpha^6 + 3^{3\beta-1}.x + x^2 \tag{5}$$

and (5) can readily be put into the form

$$4\gamma^3 = 3^{6\beta-3}\alpha^6 + (2x + 3^{3\beta-1}\alpha^3)^2 = 4\xi^2 \tag{6}$$

If ξ is an integer it can be expressed as $x + \mu$, where μ is an integer, and then it follows from (6) that

$$3^{6\beta-3}\alpha^6 = (4x + 2\mu + 3^{3\beta-1}\alpha^3)(2\mu - 3^{3\beta-1}\alpha^3) \tag{7}$$

Let us assume α is a prime for simplicity .If $(2\mu - 3^{3\beta-1}\ \alpha^3)$ is not equal to 1 , then it cannot be $3^{6\beta-3}\alpha^6$ since then $3x + x - 1 + 2\mu + 3^{3\beta-1}\alpha^3 = 0$ from which and (5) it follows that$(3,x) \neq 1$ Therefore $2\mu - 3^{3\beta-1}\alpha^3$ is equal to α^6 or $3^{6\beta-3}$ and hence

$$3^{6\beta-3} = 4x + 2.3^{3\beta-1}\alpha^3 + \alpha^6 \tag{a}$$

or

$$\alpha^{6\beta-3} = 4x + 2.3^{3\beta-1}\alpha^3 + 3^{6\beta-3} \tag{b}$$

or

$$2\mu - 3^{3\beta-1}\alpha^3 = 1 \tag{c}$$

and (c) gives

$$3^{6\beta-3}\alpha^3 = 4x + 2.3^{3\beta-1}\alpha^3 + 1 \tag{d}$$

Since $x - 1$ or $x + 1$ is divisible by 3^2 due to (5) and since $a^3 \mp b^3$ is divisible by 3^2 we conclude that (a) or (b) or (d) is never satisfied since $(3,x) \neq 1$. Hence our assumption that ξ and γ are integers never holds . If α is a composite number it can be expressed as a product of primes and the proof of the conjecture follows in the similar manner as above.

Reference:

1) Piyadasa R.A.D. & Mallawaarachchi D.K..& Munasinghe J. 10th International Conference of Sri Lanka Studies Abstract pp110, 2005

2) Piyadasa R.A.D. & Karunatileke N.G.A. 10th International Conference of Sri Lanka Studies Abstract pp 108, 2005.