A Different Look at the Primitive Integral Triads of $z^n = y^n + x^n \quad (n = 2)$ and a Conjecture on for any $n(\neq 2)$

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The primitive Pythagorean triples (x, y, z) are now well understood [1]. However, we believe that a closer look at the solution is needed along new directions to understand the terrible difficulty in giving a simple proof for the Fermat's last theorem. Keeping this fact in mind we look at the solutions of $z^2 = y^2 + x^2$, $(x, y) = 1$ in the following manner.

is a primitive Pythagorean triple if and only if

$= z^2, (x, y) = 1$

(1)It is obvious that one of (x, y, z) is even and it can be shown that z is never even by using (1) and substituting $z = y + p$, $p \geq 1$, in it.

Now either x or y is even. If we suppose that y is even, $z^2 - x^2 = y^2$ and then it follows that $z - x = 2^{2\beta - 1}$ or $z - x = 2^{2\beta - 1}\alpha^2$ where $\alpha, \beta \geq 1$ and are integers. The following are examples for the justification of our point.

$17^2 = 15^2 + 8^2 \quad, \beta = 1 \quad, z - x = 2$

$13^2 = 12^2 + 5^2 \quad, \beta = 2 \quad, z - x = 2^3$

$113^2 = 112^2 + 15^2 \quad, \beta = 1, \alpha = 7 \qquad z - x = 2 \times 7^2$

Now we apply the mean value theorem of the form

$a^2 - b^2 = 2(a - b)\xi$ where $a < \xi < b$, to the expression $z^2 - x^2$, to obtain

$z^2 - x^2 = 2.2^{2\beta - 1}\alpha^2\xi$ since $z^2 - x^2 = (z - x)(z + x)$

It follows that $z^2 - x^2 = 2.(z - x)\frac{(z + x)}{2}$

It is clear that $2.2^{2\beta - 1}\alpha^2$ or $2(z - x)$ is a perfect square and since $y^2 = 2.(z - x)\frac{(z + x)}{2}$ it follows that $\frac{z + x}{2} = \xi$ is a perfect square.

Therefore, in case of any primitive triple (x, y, z) of $z^2 = y^2 + x^2$, the mean value theorem is manifested in the form

$a^2 - b^2 = 2(a - b)\xi$ where ξ is a perfect square $b < \xi < a$.

Now we point out the following conjecture. Suppose that $z, x > n$ for any prime $n \geq 3$. Then, $z^n - x^n = n(z - x)\xi^{n-1}$ by the mean value theorem and we conjecture that ξ is irrational when $z - x = \alpha^n n^{\beta n - 1}$.

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