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Article

Solutions of Direct and Inverse Even-Order Sturm-Liouville Problems Using Magnus Expansion

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Abstract: In this paper Lie group method in combination with Magnus expansion is utilized to develop a universal method applicable to solving a Sturm–Liouville problem (SLP) of any order with arbitrary boundary conditions. It is shown that the method has ability to solve direct regular (and some singular) SLPs of even orders (tested for up to eight), with a mix of (including non-separable and finite singular endpoints) boundary conditions, accurately and efficiently. The present technique is successfully applied to overcome the difficulties in finding suitable sets of eigenvalues so that the inverse SLP problem can be effectively solved. The inverse SLP algorithm proposed by Barcilon (1974) is utilized in combination with the Magnus method so that a direct SLP of any (even) order and an inverse SLP of order two can be solved effectively.

Keywords: higher-order Sturm–Liouville problems; inverse Sturm–Liouville problems; Magnus expansion

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1. Introduction

Direct and inverse eigenvalue problems (EVP) of linear differential operators play an important role in all vibration problems in engineering and physics [1]. The theory of direct Sturm–Liouville problems (SLP) started around the 1830s in the independent works of Sturm and Liouville.The inverse Sturm–Liouville theory originated in 1929 [2].

In this paper, we consider the $2m$th order, nonsingular, self-adjoint eigenvalue problem:

$$\begin{aligned}(-1)^m(p_m(x)y^{(m)})^{(m)} + (-1)^{m-1}(p_{m-1}(x)y^{(m-1)})^{(m-1)}\\ + \ldots + (p_2(x)y'')'' - (p_1(x)y')' + p_0(x)y = \lambda w(x)y, \qquad a < x < b\end{aligned} \tag{1}$$

along with y satisfying some general, separated boundary conditions at a and b (see Equation (2)). Usually, the functions p_k, $(0 \le k \le m)$, and $w(x)$ are continuous on the finite closed interval $[a, b]$, and p_m has continuous derivative. Further assumptions A1–A4 are placed on the coefficient functions in Section 2.1. For Equation (1), the direct eigenvalue problems is concerned with determining the λ given the coefficient information p_k, $(0 \le k \le m)$, and the inverse eigenvalue problem is concerned with reconstructing the unknown coefficient functions p_k, $(0 \le k \le m)$ from the knowledge of suitable spectral data satisfying the equation.

There is a variety of numerical solutions for the simplest case of Equation (1) known as the direct Sturm–Liouville problem with $m = 1$, the most notable ones being Finite difference method (FDM) [3], Numerov's method (NM) [4], Finite element method (FEM) [5], Modified Numerov's method (MNM) [6], FEM with trigonometric hat functions using Simpson's rule (FEMS) and using