3.9 A simple analytical proof of Fermat's last theorem for n=5

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ABSTRACT

It is well known that the proof of Fermat's last theorem (FLT) for any odd prime is difficult and lengthy. Proof of FLT for $n=5$ was shared by two eminent mathematicians, Dirichlet and Lagrange[1], [2]. However, their proof is rather lengthy and difficult as well known The main objective of this contribution is to give a simple analytical proof for Fermat's last theorem for this exponent. The proof is based on three simple lemmas used in case of $n=3$.

Lemma.1

If $(a,5)=1=(b,5)$ and $a^5-b^5 \equiv 0(\mathrm{mod}\, 5^m)$, then $m \geq 2$ and $a-b \equiv 0(\mathrm{mod}\, 5^{m-1})$.

Since the proof is exactly the same as in case of $n=3$ [3], let us assume this Lemma.

Lemma.2

If the equation

$$z^5 = y^5 + x^5, (x,y)=1 \tag{1}$$

has non-trivial integer solutions for the triple (x,y,z), then one of x,y,z is divisible by 5. If one assumes that none of x,y,z is divisible by 5, taking $y=5m\pm1,\ 5l\pm2$, $x=5k\pm1,\ 5h\pm2$, it can be easily shown that $x^5+y^5 \equiv (0,\pm2,\pm6,\pm8,\pm11)(\mathrm{mod}\,5^2)$. But $z^5 \equiv (1,-1,7,-7)(\mathrm{mod}(5^2)$. Therefore we deduce that one of x,y,z is divisible by 5 since $x^5+y^5=z^5$.

It is obvious that, without loss of generality, we can assume that all x, y, z are positive. Now, either z is divisible 5 or one of x,y is divisible by 5. Then xz or yz is divisible by 5. Hence, without loss of generality, we may assume that yz is divisible by 5.

Now, let $z=y+s$. Then (1) can be written as

$$5s(y+s)y(yz+s^2) = x^5 - s^5 \tag{2}$$

Also, any prime factor of s is a factor of x^5 and s is co-prime with $y,(y+s),(yz+s^2)$. Therefore, s is a fifth power of an integer. Now, it is easy to verify that the terms $s,y,(y+s),(yz+s^2)$ are co-prime to one another. Also, $x-s=x+y-z$, which is even and co-prime with $yz+s^2$. Let $s=t^5$ and $yz=q$ to write (2) as

$$q^2 + t^{10}q - \frac{(c^5 - t^{20})}{5} = 0 \tag{3}$$

where it is assumed that $x = c.t$ .Let $c = t^4 + 5^m n$, $q = 5^m j$ .It should be noted that the identity of the Fermat's equation $x + y - z = x - t^5$ contains the factors $t, 5^m$ and other two factors of y and z as exactly in the case of Fermat's last theorem for $n = 3$ [3]. Then we can write (3) as

$$(t^{10} + 2.5^m j)^2 = [t^{20} + 4\frac{(c^5 - t^{20})}{5}] \qquad (4)$$

Now, it is easy to obtain

$$(t^{10} + 2.5^m j)^2 = [t^{10} + 2.5^m t^6 n + 2.5^{2m} t^2 n^2]^2 + 4.5^{5m-1} n^5 \qquad (5)$$

$$(j - t^6 n - 5^m t^2 n^2)(t^{10} + 5^m j + 5^m t^6 n + 5^{2m} t^2 n^2) = 5^{4m-1} n^5 \qquad (6)$$

Obviously, n can be a factor of j . Therefore it follows from (6) that

$$j - t^6 n - 5^m t^2 n^2 = 5^{4m-1} n^5 \qquad (a)$$

$$t^{10} + 5^m j + 5^m t^6 n + 5^{2m} t^2 n^2 = 1^5 \qquad (b)$$

since n and t are co-prime. From (a) and (b), we obtan

$$1^5 - t^{10} - 2.5^m (nt^6 + 5^m n^2 t^2) - 5^{5m-1} n^5 = 0 \qquad (7)$$

If $t = 1$,it follows that $n = 0$ or 5 divides n, which is a contradiction. Since all numbers in (7) are positive and $1^5 - t^{10} < 0$ when $t \neq 1$, we deduce that (7) is never satisfied. Hence, (1) has no non-trivial integer solutions.

References

(1) Paulo Ribenboim,Fermat's last theorem for amateurs, Spriger-Verlag-New York,1999.

(2) Harold M.Edwards,Fermat's Last Theorem ,Springer-Verlag-New York,1977.

(3) Piyadasa R.A.D.,(2007) .Analytical proof of Fermat's last theorem for n=3 , Annual research symposium of faculty of graduate studies, University of Kelniya,Kelaniya. 127-128.