3.3 Limits for the 'radius' of a fluid sphere for three density distributions

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ABSTRACT

The internal space time due to a sphere of fluid with radius r_0 , can be expressed as, $ds^2 = e^{\upsilon} c^2 dt^2 - \left[e^{\lambda} dr^2 + r^2\left(d\theta^2 + \sin^2\theta \; d\varphi^2\right)\right]$; where $\upsilon(r)$ and $\lambda(r)$ are functions of the radial variables which must be determined from the field equations. We have solved the field equations and obtained an expression for the interior solution of an object when the fluid sphere has constant pressure $P = \varepsilon_0 \rho_0 c^2$ and a constant density ρ_0, where ε_0 is the permittivity of free space and *c* is the velocity of light in vacuum. The solution obtained is,

$$ds^2 = \left(\frac{\hat{R}^2 + r^2}{\hat{R}^2 + r_0^{\,2}}\right)^{\left(\frac{A\hat{R}^2}{2}\right)} \left(1 - \frac{2m_0}{r_0}\right) c^2 \; dt^2 - \left[\frac{dr^2}{\left(1 - \frac{r^2}{\hat{R}^2}\right)} + r^2\left(d\theta^2 + \sin^2\theta \, d\phi^2\right)\right]$$

$$\text{where, } \hat{R}^2 = \frac{r_0^{\,3}}{2m_0} \text{ and } A = \left(\frac{3\varepsilon_0 + 1}{\hat{R}^2}\right)$$

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Then the proper volume of the sphere having the 'radius' r is given by,

$$V(r) = 2\pi\hat{R}^2\left[\sin^{-1}\left(\frac{r}{\hat{R}}\right) - \left(\frac{r}{\hat{R}}\right)\left(\left(1 - \frac{r^2}{\hat{R}^2}\right)^{\frac{1}{2}}\right)\right]$$

For the existence of the proper volume inside the solid sphere, 'radius' r should be less than or equal to the $\hat{R}$.

i.e. $r \le \hat{R}$

$$r \le \sqrt{\frac{r_0^{\,3}}{2\,m_0}}$$

When $r = r_0$ this implies

$$r_0 \ge 2\,m_0$$

The interior solution of any celestial object with constant density [(1)] obtained by Schwarzschild can be written as,

$$ds^2 = \left(\frac{3}{2}\sqrt{1 - \frac{r_0^{\ 2}}{\hat{R}^2}} - \frac{1}{2}\sqrt{1 - \frac{r^2}{\hat{R}^2}} \right)^2 c^2\, dt^2 - \left[\frac{dr^2}{\left(1 - \frac{r^2}{\hat{R}^2}\right)} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \right],\ \text{for } r \le r_{0,}\ \hat{R}^2 = \frac{3c^2}{8\pi G\rho_0}$$

The proper volume of the object having the 'radius' r is given by,

$$V(r) = 2\pi\hat{R}^2 \left[\sin^{-1}\left(\frac{r}{\hat{R}}\right) - \left(\frac{r}{\hat{R}}\right)\left(\left(1 - \frac{r^2}{\hat{R}^2}\right)^{\frac{1}{2}}\right) \right],\ \text{as in the previous case.}$$

For the existence of the proper volume inside the solid sphere 'radius' r should be less than or equal to the $\hat{R}$. i.e. $r \le \hat{R}$, where $\hat{R}$ is now $\sqrt{\frac{3c^2}{8\pi G\rho_0}}$

$$\therefore r \le \sqrt{\frac{3c^2}{8\pi G\rho_0}}$$

Since the mass of a spherical sphere of constant density is given by,

$$M = \frac{4}{3}\pi r_0^{\ 3}\rho_0$$

$$r \le \sqrt{\frac{r_0^{\ 3}}{2\,m_0}}$$

When $r = r_0$ this implies,

$$r_0 \ge 2\,m_0$$

We have also considered the case of a fluid with variable density, $\rho = \frac{\rho_0 r}{r_0}$, We find that

$$e^\lambda = \frac{1}{\left[1 - \left(\frac{r}{\hat{A}}\right)^3\right]},\ \text{where } \hat{A}^3 = -\frac{4r_0}{C\rho_0}$$

Matching the interior solution with Schwarzschild exterior solution at $r = r_0$, 'surface' of the fluid sphere ρ_0 can be expressed as $\frac{M_0}{\pi r_0^{\ 3}}$; where M_0 is the mass of the sphere as seen by an external observer.

Then the proper volume can be expressed as,

$$V(r) = \frac{8\pi\hat{A}^3}{3}\left[1 - \left(1 - \frac{r^3}{\hat{A}^3}\right)^{\frac{1}{2}}\right],\ \text{where } \hat{A}^3 = -\frac{4r_0}{C\rho_0} = \frac{r_0^{\ 4}}{2m_0}.$$

For the existence of a real proper volume inside the solid sphere the term $(1-\frac{r^3}{\hat{A}^3})$ should be non

$$1 \geq \frac{r^3}{\hat{A}^3}$$

negative. $1 \geq \frac{r^3(2m_0)}{r_0{}^4}$

$$\text{when } r = r_0 \text{ this inequality implies,}$$

$$r_0 \geq 2m_0$$

We conclude that in the case of ordinary matter applicable to the above cases the radius of the fluid sphere must be greater than the Schwarzschild radius.

References:

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