Piyadasa, R.A.D.
Department of Mathematics, University of Kelaniya
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Solution of a special Diophantine equation using elementary mathematics

Our object is to solve the Diophantine equation $x^3 = y^2 + 2$ using elementary mathematics. Fermat's solution of this equation is vague [1] ,and one has to depend on advanced mathematics such as unique factorization domain,UFD, etc. in complex number field to solve this equation completely. In the following, we solve this equation without using complex number field.

Solution: If the equation has an integer solutions for y, x, then y should be odd. If y is divisible by 3, then we must have $x^3 + 1 = y^2 + 3$ and $x^3 + 1 \equiv 0(\mod 3^2)$ and since

$$y^2 + 3 \equiv 0(\mod 3) \qquad (1)$$

y can not be divisible by 3.

Also, $x^3 = y^2 - 1 + 3$ and since $y^2 - 1$ is divisible by 3 (Fermat's little theorem).

$$x^3 - 27 = (y-5)(y+5) \qquad (2)$$

Therefore $x = 3, y = \pm 5$ is a solution. We will use a simple mathematical technique to show that there is no any other solution.

$$(x-3)[(x-3)^2 + 9(x-3) + 27] = (y-5)(y+5)$$

If $x - 3 = d$, and let

$$d\frac{n}{m} = y - 5$$

$$\frac{m}{n}(k^2 + 9k + 27) = y + 5$$

Note that m is even, $k \equiv 0(\mod 8)$, $(n,m) = 1$.

$$\frac{m}{n}(k^2 + 9k + 27) - k\frac{n}{m} = 10 \qquad (3)$$

$$k^2 - (\frac{n^2}{m^2} - 9)k - (\frac{10n}{m} - 27) = 0 \qquad (4)$$

Both solutions of this quadratic can not be integers since the sun of the roots , $\frac{n^2}{m^2} - 9$, is nit an integer.

The solutions of the equation are given by

$$k_1 = \frac{\frac{n^2}{m^2} - 9 \pm \Delta}{2} \qquad (5)$$

$$\Delta = (\frac{n^2}{m^2} - 9)^2 + 4(\frac{10n}{m} - 27) = [\frac{(n^2 - 9m^2)^2 + 4m^3(10n - 27m)}{m^4}]^{\frac{1}{2}}$$

$2m^2 k_1 = n^2 - 9m^2 + d$,where

$$\frac{(n^2 - 9m^2)^2 + 4m^3(10n - 27m)}{m^4} = \frac{d^2}{m^4} \qquad (6)$$

and we assume that $k_1 = n^2 - 9m^2 + d$ is an integer.

$$(d - n^2 + 9m^2)(d + n^2 - 9m^2) = 4m^3(10n - 27m) = 2m^2 k_1(d - n^2 + 9m^2)$$

$$2m(10n-27m)=k_1(d-n^2+9m^2)$$

Hence, the other root is also an integer contradicting our assumption that $\frac{n}{m}\neq 1$. Therefore k_1 can not be an integer. Hence, $x^3=y^2+2$ has no other solutions.

Reference & Notes

[1] Fermat and solution of $x^3=y^2+2$,J.V.Leyendekkers;A.G.Shanon, International Journal of Mathematical Education in Science andTechnology,Vol.33,Issue.1, Jan.2002, p.91-95

Solution of the Diophantine equation $x^3=y^2+2$

If the equation has an integer solution y , x should be odd. If y is divisible by 3 , then we must have $x^3+1=y^2+3$ and $x^3+1\equiv 0(\mathrm{mod}\,3^2)$ and since $y^2+3\equiv 0(\mathrm{mod}\,3)$, y can not be divisible by 3 .
Also, $x^3=y^2-1+3$ and since y^2-1 is divisible by 3 (Fermat's little theorem).

$$x^3-27=(y-5)(y+5)$$

Therefore $x=3, y=\pm 5$ is a solution. We use the same technique as before to show there is no any other solution.

$$(x-3)[(x-3)^2+9(x-3)+27]=(y-5)(y+5)$$

If $x-3=d$, and let

$$d\frac{n}{m}=y-5$$

$$\frac{m}{n}(k^2+9k+27)=y+5$$

Note that m is even, $k\equiv 0(\mathrm{mod}\,8)$, $(n,m)=1$.

$$\frac{m}{n}(k^2+9k+27)-k\frac{n}{m}=10 \qquad \text{(A)}$$

$$k^2-(\frac{n^2}{m^2}-9)k-(\frac{10n}{m}-27)=0$$

$$k_1=\frac{\frac{n^2}{m^2}-9\pm\Delta}{2}$$

$$\Delta=(\frac{n^2}{m^2}-9)^2+4(\frac{10n}{m}-27)=[\frac{(n^2-9m^2)+4m^3(10n-27m)}{m^4}]^{\frac{1}{2}}$$

$$2m^2k_1=n^2-9m^2+d$$

$$\frac{(n^2-9m^2)+4m^3(10n-27m)}{m^4}=\frac{d^2}{m^4}$$

As before $(d-n^2+9m^2)(d+n^2-9m^2)=4m^3(10n-27m)=2m^2k_1(d-n^2-9m^2)$

$$2m(10n-27m)=k_1(d-n^2-9m^2)$$

Hence, as before the other root is also an integer contradicting our assumption that $\frac{n}{m}\neq 1$. Therefore k_1 can not be an integer.